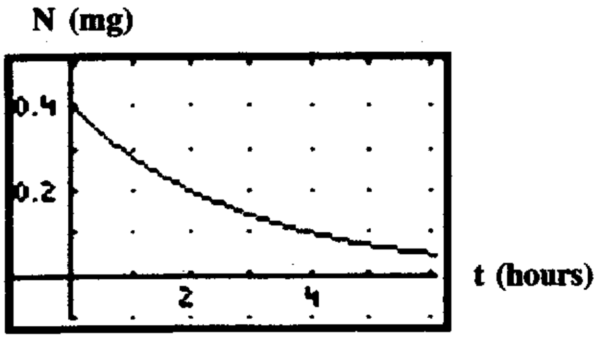
Interpreting Functions (Sections 1.1 and 1.2) **Date:** ___________

The topic of a **function** is one of the major concepts in any College Algebra course. Recall that it is used to represent the dependence of one quantity upon another. A complete understanding of functions, and function notation, is necessary to be successful in any Calculus course.

Example #1

Nicotine Problem: The graph shown below represents the amount of nicotine, $N = f(t)$, in mg, in a person's bloodstream as a function of the time t, in hours, since the person finished smoking a cigarette.



a. Estimate f(3) and interpret it in terms of the problem.

b. About how many hours have passed before the nicotine level is down to 0.1 mg? Write this result in function notation.

c. What is the vertical intercept? What does it represent in terms of nicotine?

d. If this function had a horizontal intercept, what would it represent?

Example #2

Fahrenheit/Celsius Temperatures: According to the Arizona Daily Star, the highest recorded temperature in the United States on Monday, August 17th, 2009, was 113° F (or 45° C) in Death Valley CA, and the coldest temperature was 23° F (or -5° C) in West Yellowstone, MT. Knowing that there is a linear relationship between a Celsius temperature and the corresponding Fahrenheit temperature,

a. Write an equation that shows how the Celsius temperature C depends on the Fahrenheit temperature F.

b. Interpret the slope of the line in part a above.

c. If the high temperature in San Fancisco that day was 20° C, what was the corresponding Fahrenheit temperature?

Example #3

Product Promotion: A cereal company finds that the number of people who will buy one of its products in the first month that it is introduced is linearly related to the amount of money it spends on advertising. If it spends $40,000 on advertising, then 100,000 boxes of cereal will be sold, and if it spends $60,000, then 180,000 boxes will be sold.

a. Write an equation describing the relation between the amount A spent on advertising and the number N of boxes sold.
The independent variable is: ________ The dependent variable is: ________

b. Interpret the slope of the line in part a above.

c. How much advertising is needed to sell 300,000 boxes of cereal?

Rates of Change (Section 1.3) **Date:** ____________

Last class, we wrote the equation of a line that represented the relationship between a Celsius temperature and a Fahrenheit temperature. A linear function has a constant rate of change which is the ____________________.
But, how do we calculate a rate of change for a function that is not linear?

Let's revisit the function given in the Nicotine Problem, shown below.

N (mg)

t (hours)

a. Estimate the change in the amount of nicotine over the first 6 hours. ____________________

b. Estimate the average hourly rate at which the nicotine is decreasing over the first 6 hours.
Note: The word "average" is used because, as we will see, the rate of change can vary within the interval.

c. Does it make sense that this value is negative? Explain.

d. Estimate the average rate of change of the amount of nicotine left in the body between $t = 2$ and $t = 6$.

In general, if y is a function of t, so $y = f(t)$, then the average rate of change of y between $t = a$ and $t = b$ is: Average rate of change = ____________________

Back to the Nicotine Problem.

e. What basic type of function might best represent the graph of this function? ________________

f. Find a function in the form $y = a \cdot b^t$, that could be used to model this function.

(Hint: We need to find values for a and b.) Check your answer by graphing the function on your calculator.

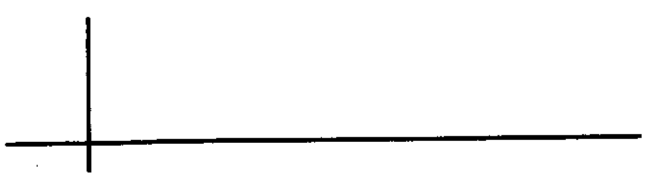
g. Use your function, not the graph, and calculate the average rate of change of nicotine between $t = 1.5$ and $t = 3.5$.

Definitions:

Example: Given the function $y = f(x) = \sqrt{x+1}$.

a. Find the average rate of change of f between $x = 0$ and $x = 3$.

b. Sketch the graph of $y = f(x)$, and represent the average rate of change as the slope of a line.

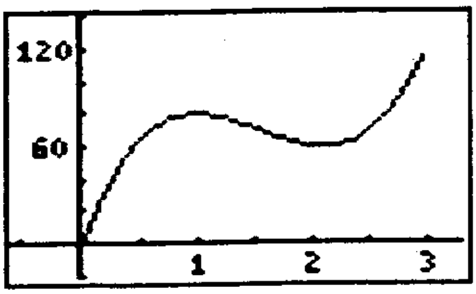


c. Which is larger, the average rate of change of the function between $x = 0$ and $x = 3$, or $x = 3$ and $x = 6$? What does this tell you about the graph of the function?

Definitions:

Examples:

1. Given the graph of the function below.



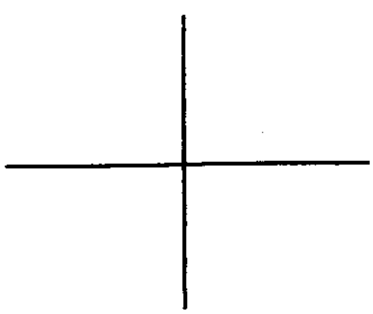
Estimate the interval(s) over which the function is:

a. Increasing ______________________ b. Decreasing ______________________

c. Concave Up ______________________ d. Concave Down: ______________________

e. Estimate the average rate of change between t =1 and t = 3. ______________________

2. Draw a circle below with center at the origin.



Identify a section (quadrant) of the circle that is:

a. decreasing and concave up. ____________ b. increasing and concave down. ____________

c. increasing and concave up. ____________ d. decreasing and concave down. ____________

3. Values of 5 functions, F(t), G(t), H(t), I(t), and J(t) are shown in the chart below.

t	F(t)	G(t)	H(t)	I(t)	J(t)
10	15	15	45	60	15
20	22	18	36	57	17
30	28	21	30	52	20
40	33	24	26	44	24
50	37	27	23	33	29

Identify the function whose graph would be:

a. linear. ____________

b. decreasing and concave up. ____________

c. increasing and concave down. ____________

d. increasing and concave up. ____________

e. decreasing and concave down. ____________

Distance, Velocity, and Speed

A ball is thrown up in the air. The height, y, of the ball above the ground first increases, and then decreases. It's height t seconds after it is thrown is shown in the chart below.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4
y (feet)	5	36	59	74	81	80	71	54	29

1. What is the change in the ball's height during the first 3 seconds? ____________________

2. What is the average rate of change of the height of the ball during the first 3 seconds? ____________________

Note: The average rate of change of height with respect to time is the ______________________________

3. Find the average velocity of the ball over the time interval from t = 2 to t = 4. Explain the sign of the answer.

Note: There is a difference between <u>velocity</u> and <u>speed</u>. Speed is the <u>magnitude of velocity</u>, so it is always positive.

Instantaneous Rate of Change (Section 2.1) **Date:** ____________

Last class period, we looked at the problem where a ball is thrown up in the air. The height, y, of the ball above the ground first increases, and then decreases. It's height t seconds after it is thrown is shown in the chart below.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4
y (feet)	5	36	59	74	81	80	71	54	29

Our objective was to calculate the average rate of change (average velocity) of the ball over an interval.

Example: Calculate the average rate of change of the ball on the interval from $t = 0.5$ to $t = 3.5$.

A new question: What is the velocity of the ball at exactly 1.5 seconds? This would be the instantaneous velocity at $t = 1.5$.

We can use the **average velocities** to estimate the **instantaneous velocity**. How?

Suppose we knew the function rule that generated the values in the chart above. Enter the data into the lists of your calculator and create a scatterplot.

What type of function best fits this data? ______________________________

Then using the data, perform a **quadratic regression** (QuadReg under STAT, CALC) to find the best fitting quadratic function for the data.

y = ____

How could we use this function to find a better estimate for the instantaneous velocity at $t = 1.5$ seconds?

From the calculations above, it appears that the instantaneous velocity at $t = 1.5$ seconds is exactly ____________
This value is the "limiting value" or the "limit" of the average velocities over smaller and smaller intervals.

Definitions:

1. The **instantaneous velocity** of an object at time t is defined to be ____

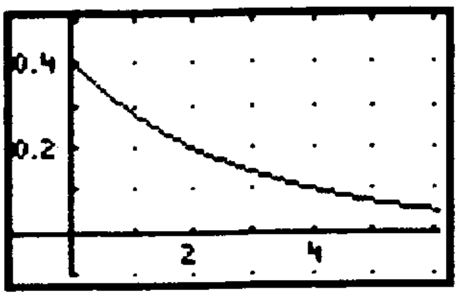
2. The **instantaneous rate of change of a function f at a**, also called the rate of change of f at a, is defined to be

____

Example:

Recall the **Nicotine Problem** from a few days ago.

Nicotine Problem: The graph shown below represents the amount of nicotine, $N = f(t)$, in mg, in a person's bloodstream as a function of the time t, in hours, since the person finished smoking a cigarette.



From the graph, we determined that the function rule that represented the problem was: ____________________

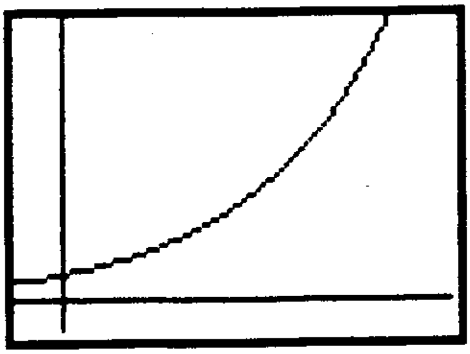
Use the function rule to estimate the instantaneous rate of change of N with respect to t when $t = 1$. ____________

The **instantaneous rate of change** of a function is so important, that it is given it's own name.

The **derivative of f at a,** written $f'(a)$ (read "f-prime of a), is defined to be the **instantaneous rate of change of f at the value x = a.**

Example: If $f(x) = x^3$, estimate $f'(2)$.

Visualizing the Derivative Graphically



Notes:

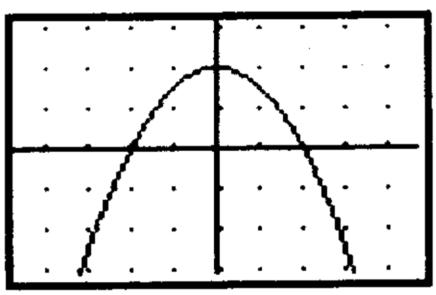
The **average rate of change** of a function is represented by the slope of ______________________

The **instantaneous rate of change** of a function is represented by the slope of ______________________

The following three statements are equivalent concerning a function f and an x-value of x = a.
1. ______________________
2. ______________________
3. ______________________

Example:

The graph of $f(x) = -\frac{1}{2}x^2 + 2$ is shown to the right.



1. Use the graph to determine whether each of the following quantity is positive (+), negative (-), or zero (0).

a. f(1) ________ b. f(-3) ________ c. f(0) ________ d. f'(1) ________

e. f'(-3) ______ f. f'(0) ________ g. $\dfrac{f(1)-f(-1)}{1-(-1)}$ ________

2. Which is larger, f(-3) or f(-1)? ______________

3. Which is larger, f'(-3) or f'(-1)? ______________

4. Estimate f'(2) graphically and numerically. Graphically: ________ Numerically: ________

The Derivative Function Graphically (Section 2.2) **Date** ____________________

In the last class, we looked at the derivative of a function at a point. In general, the derivative takes on different values at different points and is itself a function.

Remember, **the derivative f'(a) is the slope of the tangent line to the graph of f at x = a.**

One question in this lesson is:

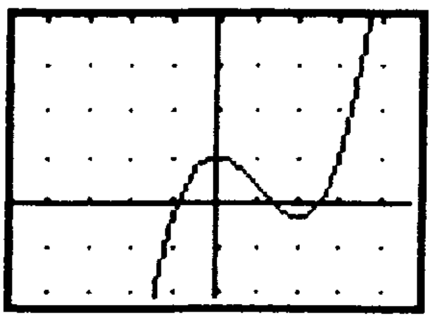
Given the graph of a function f, can we sketch a graph of the derivative function f'?

And another is:

What does the derivative f' tell us about f?

We are going to try to answer these questions, because, if we can, we should have a good understanding of the derivative function!

Example: Below is the graph of the function $f(x)=\frac{1}{3}x^3-x^2+1$.



1. At what x-value(s) does it appear that the derivative of f is zero? x = ______________________________

2. From the graph, estimate the value of f'(1). ____________________ f'(-1) ____________________

3. Built into your calculator is a feature that will estimate the derivative of a function at a value. From the Home screen of your calculator, press the MATH key, and select **8:nDeriv(**. (This stands for a "numerical derivative".) The parameters for "nDeriv" are:

nDeriv(function , variable , value)

So, if we want the derivative of $f(x)=\frac{1}{3}x^3-x^2+1$, at x = 0, we would enter:

____

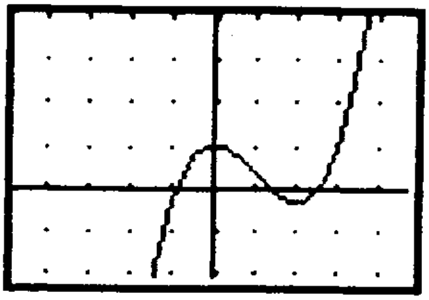
Check your other answers from parts 1 and 2 above, and then complete the chart below.

x	-2	-1	0	1	2	3
f'(x)						

An important point to notice is that for every x-value, there is a corresponding value of $f'(x)$. The derivative, therefore, is also a function of x!

For a function f, we define the **derivative function,** f', as the instantaneous rate of change of f at x.

On the graph of the function $f(x)=\frac{1}{3}x^3-x^2+1$ below, plot the points (x , f'(x)) that you can fit on the graph from the table on the previous page. Then connect the points with a smooth curve.



Can you guess the rule for the derivative function from its graph?

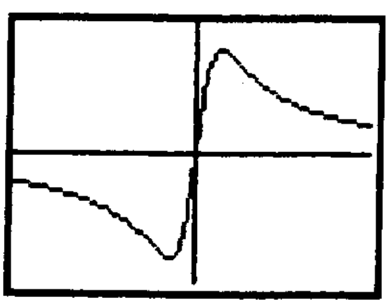
Guess: f'(x) = ______________________

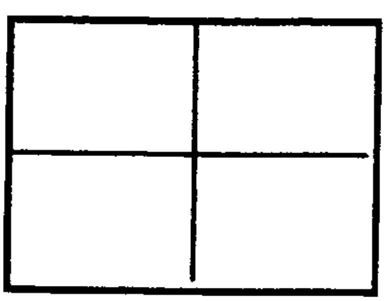
What properties can we write about the relationship between the graph of f and the graph of f'?

1. Where f has a "turning point", ______________________
2. Where f is increasing, ______________________
3. Where f is decreasing, ______________________

Examples: Given the graph of the function shown, sketch the graph of the derivative function f' directly below it. Remember, **"the y value on the graph of f' is the slope of the tangent line to the graph of f."**

1.





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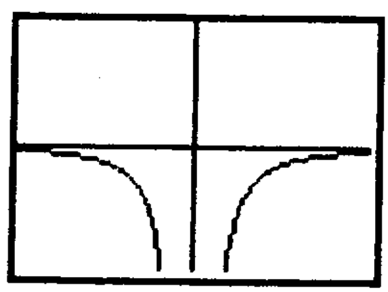
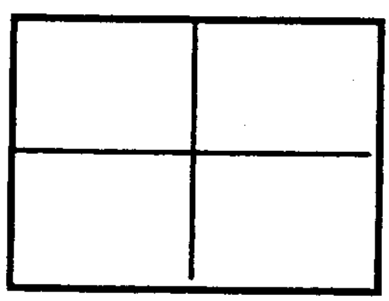
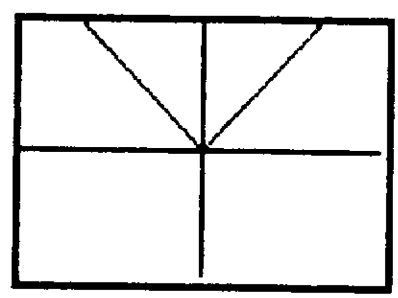
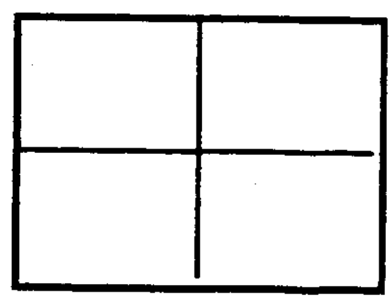




3. .

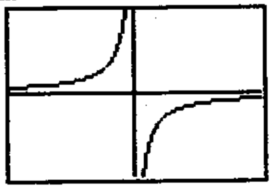




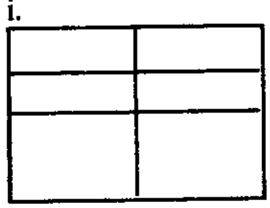
Graphs of f and f'

1. In the left column below are graphs of several functions. In the right-hand column - in a different order - are graphs of the associated <u>derivative</u> functions. Match each function with its derivative. (Note: The scales on the graphs are not all the same.)

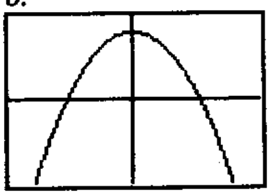
a.



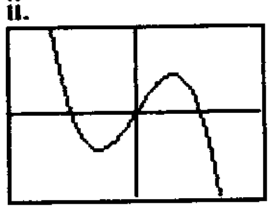
i.



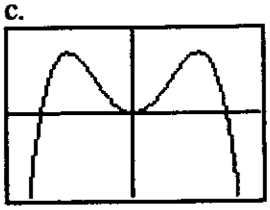
b.



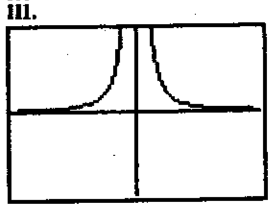
ii.



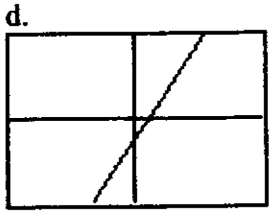
c.



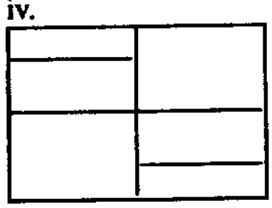
iii.



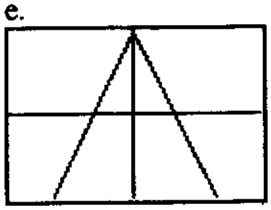
d.



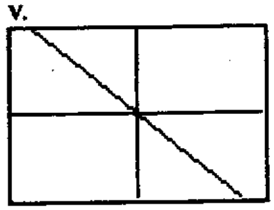
iv.



e.



v.

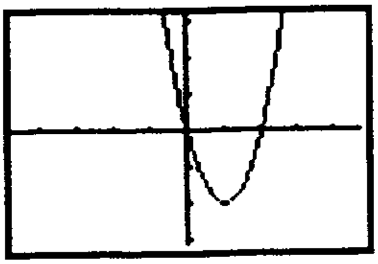


2.(a) Sketch a graph of the <u>derivative</u> of each function labeled (i) - (v) in the right column of the preceding problem.

(b) (Optional!) For each function labeled (a) - (e) in the left column of the preceding problem, sketch a graph of a function whose <u>derivative</u> is the function shown.

The second question we wanted to answer was: **What does the derivative f' tell us about f?**

Examples: Below is the graph of f', the derivative of a function f.

1. 

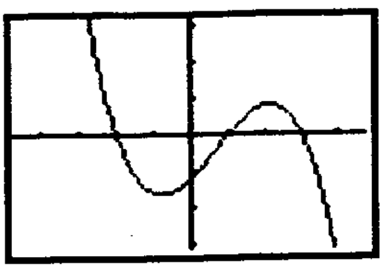
On what interval(s) is the function f

a. increasing? ______________________

b. decreasing? ______________________

At what x-value does f have a

c. maximum? __________ d. minimum? __________

2. 

On what interval(s) is the function f

a. increasing? ______________________

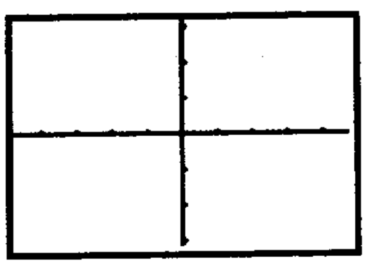
b. decreasing? ______________________

At what x-value does f have a

c. maximum? __________ d. minimum? __________

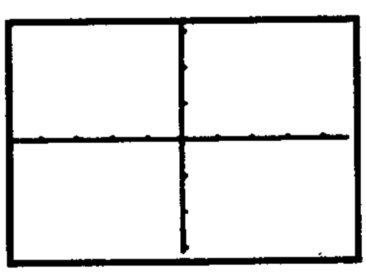
3. Draw a possible graph of $y = f(x)$ given the following information.

- $f(1) = 2$
- $f'(x) > 0$ on $x < 1$
- $f'(x) < 0$ on $x > 1$



4 Draw a possible graph of $y = f(x)$ given the following information.

- $f(-2) = 0$
- $f'(0) = 0$
- $f'(x) < 0$ on $x < 0$
- $f'(x) < 0$ on $x > 0$



Interpretations of the Derivative (Section 2.3) **Date** ______________

There is an alternative notation for the derivative that we have to be familiar with.

We know that the derivative $f'(x)$ is approximated by the average rate of change over a small interval. Therefore, if $y = f(x)$, then the average rate of change is given by $\Delta y / \Delta x$. And, for "small" Δx, we have:

$$f'(x) \approx \frac{\Delta y}{\Delta x}.$$

To remind us of this, if $y = f(x)$, we can write; $f'(x) = \dfrac{dy}{dx}$.

This is known as writing the derivative using "Leibniz notation".

A disadvantage of this notation is how we need to write the "derivative evaluated at a number". Using this alternative notation, in order to write $f'(5)$, we have to write $\left.\dfrac{dy}{dx}\right|_{x=5}$.

Example: Recall that if $s = f(t)$ represents the position of a moving object at time t, then $v = f'(t)$ is the velocity of the object at time t. Using Leibniz notation we can write:

$v =$ ________________ and $f'(3) =$ ________________

Also, since the "derivative of velocity" is the "rate of change of velocity", then $\dfrac{dv}{dt}$ represents

______________________________. If the units for the velocity are meters/sec, then the units for

acceleration would be: ____.

So, a new and important question! **What is the meaning of the derivative in a real world situation?** We will try to answer this question through some examples.

Examples:

1. The population of the world, P in billions of people, is a function of the year t. Therefore, $P = f(t)$. Explain:

   a. $f(1990) = 5.295$

   b. $f'(1990) = 0.086$

   c. Use this information to estimate the world population in 1991.

2. The cost, C (in dollars) to produce g gallons of ice cream can be expressed as $C = f(g)$. Using units, explain the meaning of the following statements in terms of ice cream.

a. $f(200) = 350$

b. $f'(200) = 1.4$

c. Estimate $f'(199)$

3. The table below shows world gold production, $G = f(t)$, as a function of the year, t.

t (year)	1990	1993	1996	1999	2002
G (mn troy ounces)	70.2	73.3	73.6	82.6	82.9

a. Does $f'(t)$ appear to be positive or negative? ____________ This is because the gold production is ______________________________

b. In which time interval does $f'(t)$ appear to be the greatest? ______________________

c. Estimate $f'(2002)$. Give units and interpret your answer in terms of gold production.

d. Use the estimated value of $f'(2002)$ to estimate $f(2003)$ and $f(2010)$.

$f(2003) \approx$ ______________ $f(2010) \approx$ ______________________________

4. The table below shows a function $f(t)$, the total sales of music compact discs (CDs), in millions.

t (year)	1994	1996	1998	2000	2002
CD sales	661.2	778.9	847.0	942.5	803.3

a. On what interval(s), does $f'(t)$ appear to be positive? ______________ negative? ____________

b. Estimate $f'(2002)$. Give units and interpret your answer your answer.

c. Use the estimated value of $f'(2002)$ to estimate $f(2003)$ and $f(2010)$.

$f(2003) \approx$ ______________ $f(2010) \approx$ ______________________________

The Second Derivative **(Section 2.4)** **Date** ________________

Since the derivative is itself a function, we can calculate its derivative. This is called the **second derivative** and, as we will see, it also gives us useful information about the original function.

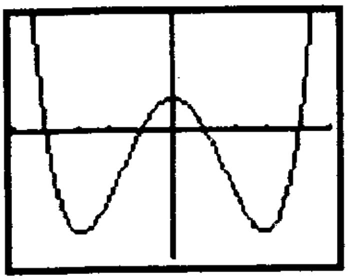
First, let's look at some notation. Let $y = f(x)$.

First derivative: Second derivative:

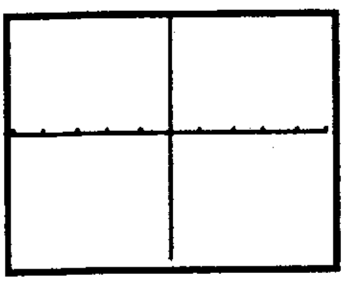
Question: What information does the second derivative tell us?

Let's look at this graphically with an example.

This is the graph of a function f.



Sketch the graph of the derivative of f, f'.



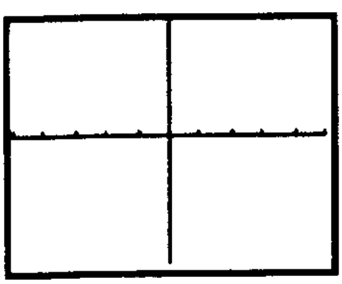
Recall the information that the derivative tells us about a function.

On an interval,

- if $f' > 0$, then ________________________________

- if $f' < 0$, then ________________________________

Sketch the graph of the derivative of f', f''.



Since f'' is the derivative of f', we have:

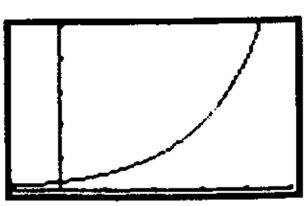
On an interval,

- if $f'' > 0$, then ________________________________

- if $f'' < 0$, then ________________________________

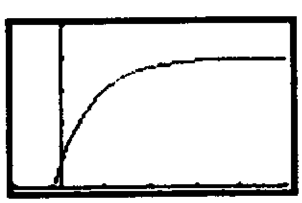
So, the question now becomes: **What does it mean for f' to be increasing or decreasing?**

Look at the graphs of four functions below. For each graph, determine if f' is **increasing** or **decreasing**? Remember, f' represents the <u>slope of the curve</u> (or the <u>slope of the tangent line</u>).



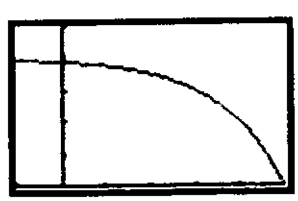
f' is ______________

f is ______________



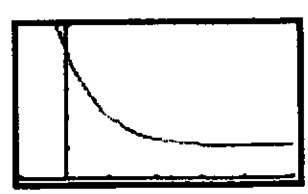
f' is ______________

f is ______________



f' is ______________

f is ______________



f' is ______________

f is ______________

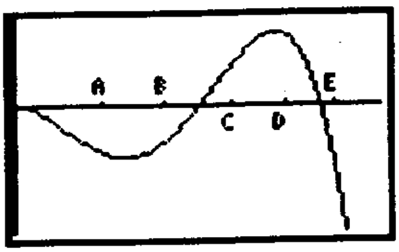
Conclusions: On an interval,

$f'' > 0$ means f' is ____________________, so f is ____________________

$f'' < 0$ means f' is ____________________, so f is ____________________

Examples:

1. Given the graph of a function f below. At which marked x-value(s) are the following statements true?



a. $f(x) < 0$ __________

b. $f'(x) < 0$ __________

c. $f''(x) < 0$ __________

d. $f(x)$ is inc __________

e. $f'(x)$ is inc __________

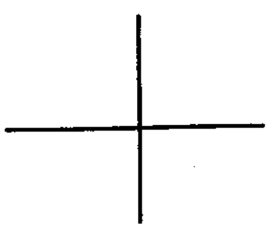
2. Let P(t) represent the price of a share of stock of a corporation at time t. What do each of the following statements tell us about the signs of the first and second derivatives of P(t)?

a. " The price of the stock is rising faster and faster". $P'(t)$ __________ $P''(t)$ __________

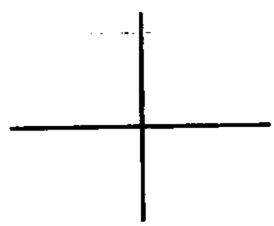
b. "The price of the stock is close to bottoming out". $P'(t)$ __________ $P''(t)$ __________

3. Sketch a graph of a continuous function with the following properties.

a. $f(0) = 1,\ f'(0) = -2,\ f''(0) < 0$



b. $f(0) = -1,\ f'(0) = 1,\ f''(-1) > 0, f''(1) < 0$



Limits and the Derivative Function Date: ______________

Earlier we defined f'(a), the derivative at x = a, graphically as:

An <u>alternative definition</u> for f'(a) could be:

Both of these definitions rely on an understanding of a **limit**. So let's develop the concept of a limit. It's important that we understand this concept!

Examples

1. Let f(x) = 2x + 1. As x gets closer and closer to some number, say 3, does f(x) get closer and closer to some value L? If it does, then we write $\lim_{x \to 3}(2x+1) = L$.

Let's see. Evaluate: f(2.9) = ________ f(2.99) = ________ f(2.999) = ________

f(3.1) = ________ f(3.01) = ________ f(3.001) = ________

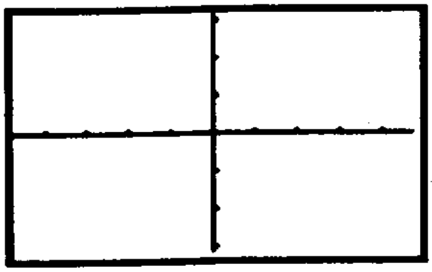
So, the $\lim_{x \to 3}(2x+1) =$ ______

How else could we have evaluated this limit?

2. Find $\lim_{x \to 2} \frac{x-2}{x^2-2x}$. Can we find the limit by substituting 2 for x? ________ Explain!

Let's look at this problem graphically. Graph the function $f(x) = \frac{x-2}{x^2-2x}$ on your calculator.

Use a "Decimal window" by selecting ZOOM, 4:ZDecimal.



What do you notice about the graph?

____

Evaluate the function close to x = 2 to determine the limit.

Conclusion: $\lim_{x \to 2} \frac{x-2}{x^2-2x} =$ ________

How could we have evaluated this limit analytically?

$$\lim_{x \to 2} \frac{x-2}{x^2-2x} =$$

Examples:

1. Graphically, find the following limit. Write your answer to 3 decimal places.
(Note: The answer to this question is a very important number. Do you remember what it is?)

$$\lim_{x \to 0} (1+x)^{1/x} = \_\_\_\_\_\_\_\_\_\_\_\_$$

2. Analytically, find the following limit.

$$\lim_{x \to -3} \frac{x^2-9}{2x^2+5x-3}$$

Let's go back to our new definition of a derivative for the next example.

Definition: $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Example: If $f(x) = 2x^2 + 3x$, find $f'(-2)$.

Practice Problem: If $f(x) = -x^2 - 4$, use the definition of the derivative to show that $f'(-3) = 6$.

Since we know that the derivative of a function f(x) is another function f'(x), we can find this derivative function if we substitute the variable **x** for the value **a** into the derivative definition. This gives us the **definition of the derivative function f'(x).**

Given a function **f(x)**, the derivative **f'(x)** is defined to be:

f'(x) =

Example: If $f(x) = 2x^2 + 3x$, use the definition of the derivative to find f'(x).

Practice Problem: If $f(x) = -x^2 - 4$, use the definition of the derivative to show that $f'(x) = -2x$.

Example: If $f(x)=\frac{1}{x^2}$, use the definition of the derivative to find f'(x).

Practice Problem: If $f(x)=\frac{1}{x}$, use the definition of the derivative to show that $f(x)=-\frac{1}{x^2}$.

Derivative Properties for Powers and Polynomials (Section 3.1) **Date:** ____________

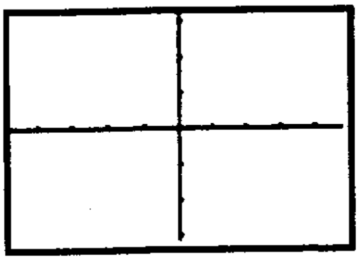
We know that the derivative of a function at a point represents a slope and a rate of change. In the last chapter we learned how to estimate values of the derivative of a function given by a graph, a table or a function. Now we learn how to find a formula for the derivative function if we are given a function rule.

We will discover these properties using a graphical approach. Some with the help of our calculator.

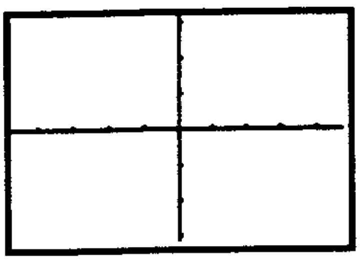
I. The Derivative of a Constant Function

1. Let $f(x) = 2$. Graph $f(x)$ and $f'(x)$.

f

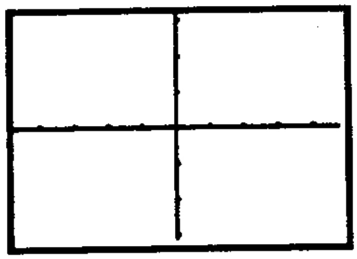


f'

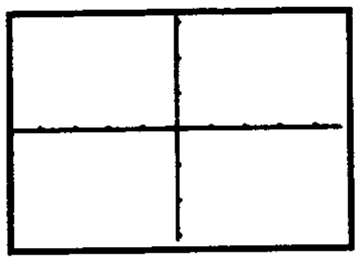


2. Let $f(x) = -1$. Graph $f(x)$ and $f'(x)$.

f



f'

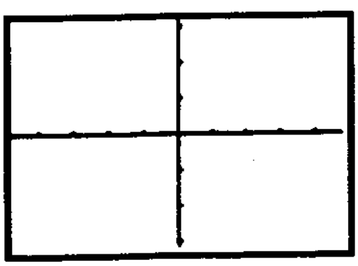


Property: **If k is a number and $f(x) = k$, then $f'(x) =$** ____________

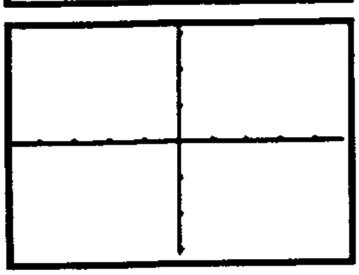
II. The Derivative of a Linear Function

1. Let $f(x) = \frac{1}{2}x - 2$. Graph $f(x)$ and $f'(x)$.

f

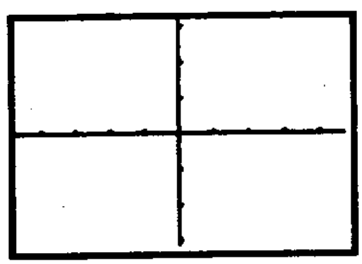


f'

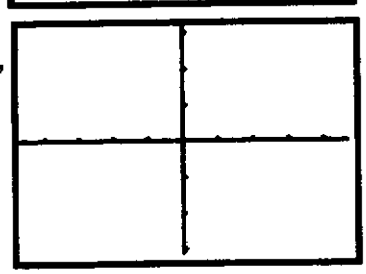


2. Let $f(x) = -3x + 3$. Graph $f(x)$ and $f'(x)$.

f



f'



Property: **If $f(x) = mx + b$, then $f'(x) =$** ____________

III. The Derivative of a Power Function ($f(x) = x^n$)

We are going to use our calculator to help us to discover this rule and other rules. For each of the following functions:

a. Set a WINDOW on your calculator of [-4 , 4] by [-15 , 15]
b. Put the given function **f(x)** into **Y1** of your graphing calculator. (You may want to turn it off by deactivating it.)
c. Let **Y2 = nDeriv(Y1,X,X)**.
d. Guess the function that you see in **Y2** and check your guess by putting it into **Y3**.
e. If it matches, record your answer; if it doesn't, try again!
f. Don't forget, we are looking for patterns and generalizations that we can write as a property.

1. $f(x) = x^2$ $\quad f'(x) =$ ____________

2. $f(x) = x^3$ $\quad f'(x) =$ ____________

3. $f(x) = x^4$ $\quad f'(x) =$ ____________

__Property:__ If $f(x) = x^n$, then $f'(x) =$ ____________________ (This is called the **Power Rule.**)

IV. The Derivative of a Constant Times a Function

1. $f(x) = 4x^2$ $\quad f'(x) =$ ______________________

2. $f(x) = -2x^3$ $\quad f'(x) =$ ______________________

3. $f(x) = 7x$ $\quad f'(x) =$ ______________________

__Property:__ If k is a number and $f(x) = k \cdot g(x)$, then $f'(x) =$ ______________________

V. The Derivatives of Sums and Differences

1. $f(x) = -x^2 - 3x - 6$ $\quad f'(x) =$ ______________________________

2. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6,$ $\quad f'(x) =$ ______________________________

__Property:__ If $f(x) = g(x) + k(x)$, then $f'(x) =$ ______________________________

VI. Question: Does the Power Rule hold for other numbers besides positive whole numbers? How about negative whole numbers? How about rational numbers?

> Let's do a **Quick Algebra Review** of exponents first.
>
> Rewrite each expression without negative or rational exponents.
>
> 1. $2^{-3} =$ ________________ 2. $x^{-n} =$ ________________
>
> 3. $16^{\frac{1}{4}} =$ ________________ 4. $x^{\frac{1}{n}} =$ ________________
>
> 5. $27^{\frac{4}{3}} =$ ________________ 6. $x^{\frac{m}{n}} =$ ________________

1. $f(x) = \dfrac{1}{x} =$ ______________, $f'(x) =$ ______________________________

2. $f(x) = \dfrac{1}{x^2} =$ ______________, $f'(x) =$ ______________________________

3. $f(x) = \sqrt{x} =$ ______________, $f'(x) =$ ______________________________

4. $f(x) = \sqrt[3]{x^2} =$ ______________, $f'(x) =$ ______________________________

VII. Practice Problems

Find the derivative of each function below. (Hint: Rewrite f first using properties of exponents!)

1. $y = 2x^4 + 2 + \dfrac{1}{2x^4}$

2. $y = 4\sqrt{x} + \dfrac{4}{\sqrt{x}} + \dfrac{\sqrt[4]{x}}{4}$

3. $y = \sqrt{x}\left(x^2 - \dfrac{1}{x^2}\right)$

VIII. Using the Derivative Formulas

1. Let $f(x) = -2x^3 + 6x + 8$

   a. Find $f'(2)$

   b. Find $f'(-1)$

   c. Find the equation of the tangent line at $x = -2$.

   d. Find the x-value(s) where the tangent line to the curve is horizontal.

   e. Check your answers to parts a - d above graphically.

2. Earlier in the semester, we looked at the problem where a ball is thrown up in the air. The height, y, of the ball above the ground first increases, and then decreases. It's height t seconds after it is thrown is shown in the chart below.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4
y (feet)	5	36	59	74	81	80	71	54	29

Our objective was to calculate the average rate of change (average velocity) of the ball over an interval and to estimate the instantneous velocity at a specific time. We entered the data into the lists of our calculator, and performed a quadratic regression to find the best fitting quadratic function for the data. The result was:

$$s(t) = -16t^2 + 70t + 5$$

a. Find the velocity function v(t), which will allow us to calculate the velocity of the ball at any time.

$v(t) =$ ____________________

b. What was the velocity of the ball after 2 seconds? ____________________

c. At what time did the ball reach it's maximum height? $t =$ ____________________

d. What is the maximum height of the ball? ____________________

e. What was the velocity of the ball when it hit the ground? Solve this algebraically, and check your answer graphically.

$v =$ ____________________

Exponential Functions - An Algebra Review (Section 1.5) Date ______________

After the linear function, the exponential function is the most important function for the application of mathematics to real world problems.

Let's begin with an example.

Example. Two situations are described below. For each situation, complete the table of values of the population P of the town at the indicated number of years t, and then find a function P(t) for the population of the town in terms of the years t.

a. The town has a population of 1000 in year 0, and grows at a rate of 50 people per year.

Years	0	1	2	3	4	Formula
Population						P(t) =

b. The town has a population of 1000 in year 0, and grows at a rate of 5% per year.

Years	0	1	2	3	4	Formula
Population						P(t) =

The function we have discovered in part a is a ______________________ function.

The function we have discovered in part b is an ______________________ function. The function tells us that the initial value of the function (when $t = 0$) was ______________, the rate of growth was __________ (or __________ %) and the growth factor was ______________.

Graph the two functions on your calculator to see how they compare.

Example: Look at the values in the following table

x	0	1	2	3	4
f(x)	1875	1125	675	405	243

Do the values in the table represent a linear function? Why or why not?

Do the values in the table represent an exponential function? Why or why not?

Find a function for the values in the table. f(x) = ______________________. Sketch a graph of f on your calculator. Why does this graph look different than the exponential function in the earlier example?

This is an example of <u>exponential decay</u>, not <u>exponential growth</u>. We can tell because the base of the exponential function is ________________________. The rate of decay is ____________ or ____________%, and the decay factor is ________________.

The General Exponential Function: We say that P is an exponential function of t with base a if:

$$P = P_0 a^t,$$

where P_0 is the ______________________________________ (when $t = 0$), and a is the ______________________________________, that is, the factor that P changes when t increases by 1. We have **exponential growth** when __________, and **exponential decay** when __________.

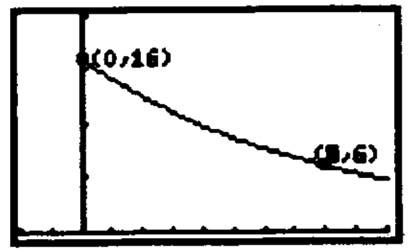
The factor a is given by:

$$a = 1 + r$$

where r is the decimal representation of the percent rate of change. If $r > 0$, then we have ______________________________ and if $r < 0$, we have ______________________________

Examples:

1. Find an exponential function in the form $f(x) = a \cdot b^x$ for the graph shownto the right. (Round the value of b to 3 decimal places.)



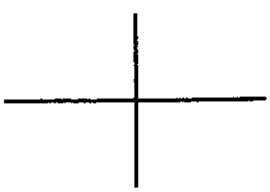
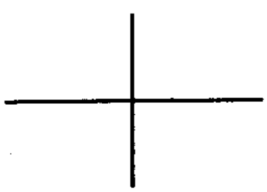
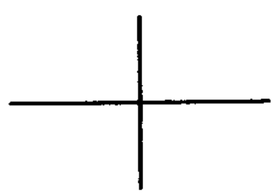
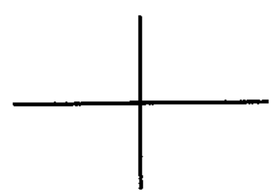
$f(x) =$ ____________________________

This function has a growth/decay rate of ________________

2. The company that produces Cliff Notes (the abridged versions of classic literature) was started in 1958 with \$4000 and sold in 1998 for \$14,000,000. Find the annual percent increase in the value of this company over the 40 years, and write a function for the value of the Cliff Notes company in terms of the years since 1958.

3. Without your calculator, sketch the graph of the following "series of functions". Label the y-intercept and the horizontal asymptote for each. Check your answers with your calculator.

a. $y = (0.6)^x$ b. $y = 8(0.6)^x$ c. $y = -8(0.6)^x$ d. $y = -8(0.6)^x + 15$

You may remember (hopefully), from your previous algebra course, that there is a very important base when studying exponential functions. In fact, this is the most commonly used base when using exponential functions.

This base is the number __________, which is approximately equal to ________________.

Since this number is between 2 and 3, the graph of $y = e^x$ should be between $y = 2^x$ and $y = 3^x$. Try it!

The base e is used so often that it is called the <u>natural base</u>, and the function $y = e^x$ is called the <u>natural exponential function</u>. (We will get an indication of "why" when we apply calculus to the exponential function!)

The Natural Logarithmic Function - An Algebra Review (Section 1.6) **Date:** ____________

We know how to algebraically solve equations like $400 = 5 \cdot x^7$.

$x \approx$ ________

But how about an equation like $400 = 5 \cdot 7^x$? In this equation the variable x is an exponent, and therefore, it is called an **exponential equation.** Solving such an equation requires an understanding of logarithms.

We'll come back to this problem.

> **Definition of the Natural Logarithm.**
>
> The **natural logarithm of x,** written **ln x** is defined as: **ln x = c** means ________________.
>
> In other words, **ln x** is the ________________________________.
>
> Note: The only logarithm we will be using in this course is the natural logarithm.

Using this definition, evaluate the following (without a calculator).

1. $\ln 1 =$ ________ because ________________
2. $\ln e =$ ________ because ________________
3. $\ln\left(\frac{1}{e}\right) =$ ________ because ________________
4. $\ln\sqrt{e} =$ ________ because ________________
5. $\ln(-10) =$ ________________ because ________________________________
6. $\ln 0 =$ ________________ because ________________________________

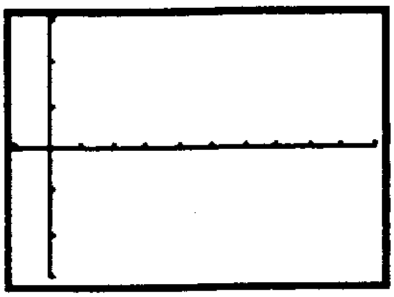
You may also remember (hopefully), some properties of logarithms.

1. $\ln(A \cdot B) =$ ________________________
2. $\ln\left(\frac{A}{B}\right) =$ ________________________
3. $\ln(A^n) =$ ________________________
4. $\ln e^x =$ ________________________
5. $e^{\ln x} =$ ________________________

Example: Apply the properties of logarithms to the expression $\ln\left(\frac{3\sqrt{x}}{yz^4}\right)$ in terms of ln x, ln y, ln z.

$$\ln\left(\frac{3\sqrt{x}}{yz^4}\right) =$$

Using your calculator, sketch a graph of $f(x) = \ln x$, and answer the following questions concerning the function.



1. State the Domain: ____________ Range: ____________

2. What's the x-intercept? ____________

3. When is $f(x) > 0$? __________ $f(x) < 0$? ____________

4. When is $f'(x) > 0$? __________ $f'(x) < 0$? ____________

5. When is $f''(x) > 0$? __________ $f''(x) < 0$? ____________

Now, let's go back to the example of solving the exponential equation $400 = 5 \cdot 7^x$.

Examples:

1. Last class, the function that we developed to solve the Cliff Notes problem was $P(t) = 4000(1.226)^t$, where t represented the number of years after 1958. In what year did the value of the Cliff Notes company reach 5 million dollars?

2. Solve the equation: $24 = 5e^{3t}$.

Exponential Functions with Base e

Last class period we saw that an exponential function with base "a" has a formula

$$P = P_0 a^t.$$

Recall that P_0 represents ____________________, and a represents the ____________________.

If $a > 1$, then we have ____

If $0 < a < 1$, then we have ____

This model is generally used when the growth rate is measured "per unit of time"; for example, a population is increasing 5% per year.

If, however, a quantity is increasing (or decreasing) at a **continuous growth rate,** the function is usually written in the form

$$P = P_0 e^{kt}.$$

What we have done, is replaced the "a" with an "e^k". Let's see what effect this has.

1. Let $a = e^k$, Solve for k. $k =$ ____________

2. If $a > 1$, then k __________, and if $0 < a < 1$, then k ______________.
3. The value of k is called the **continuous growth rate.**

Examples:

1. Convert the function $P = 700(1.07)^t$ to the form $P = P_0 e^{kt}$. Interpret the results.

2. Convert the function $P = 2000e^{-0.04t}$ to the form $P = P_0 a^t$. Interpret the results.

The Derivatives of Exponential Functions **(Section 3.2)** **Date:** ______________

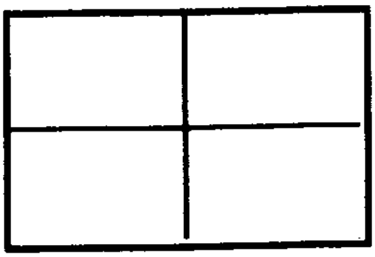
Quick note about logs. We know: $\ln 1 =$ __________, $\ln e =$ __________, $\ln 2 =$ __________

Today we want to discover the derivatives of exponential functions. So, what do we expect the derivative of a function in the form $f(x) = a^x$ to look like?

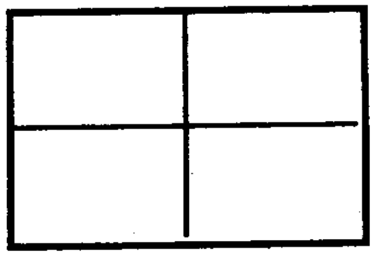
First, we know $f(x) = a^x$ has two different basic shapes. One, if $a > 1$, and the other if $0 < a < 1$.

Sketch the two graphs below.

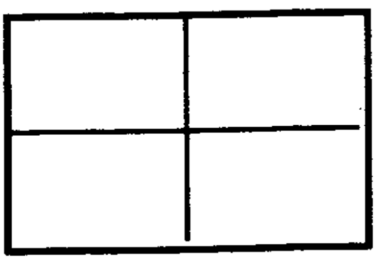
$a > 1$

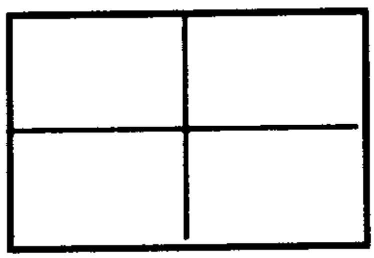


$0 < a < 1$



Sketch the graph of the derivative function of each of the above.





So, if $a > 1$, the graph of f' resembles the graph of f itself! Is it the same? Let's see by examining the graphs of $f(x) = 2^x$ and $f(x) = 3^x$, and their derivatives.

As we have done before, in your calculator enter:

$Y1 = 2^X$ and $Y2 = \text{nDeriv}(Y1, X, X)$

What are your observations? ______________________________

Now change Y1 to $Y1 = 3^X$. (We don't have to change Y2.)

What are your observations? ______________________________

This leads to two important questions.

1. Is there a number between 2 and 3 for the base of the exponential function that will make the graph of the function f and the function f' match exactly?

2. What are the exact derivatives of $f(x) = 2^x$ and $f(x) = 3^x$?

Let's look at question #1 first.

1. Is there a number between 2 and 3 for the base of the exponential function that will make the graph of the function f and the function f' match exactly?

What do you think?

Conclusion: If $f(x) =$ ______________ then $f'(x) =$ ______________

What does this say about the graph of $f(x) = e^x$? ____

____

Now question #2.

2. What is the exact derivatives of $f(x) = 2^x$ and $f(x) = 3^x$?

Again, let Y1 = 2^X. (Keep Y2 = nDeriv(Y1 , X , X))

It appears that $\frac{d}{dx}\left(2^x\right) =$ ________________________ TRACE on the derivative function to find the constant k.

It seems that k = ______________. We have seen this number before. Where? ______________________________

Conclusion: $\frac{d}{dx}\left(2^x\right) =$ ________________________

Make a conjecture about the derivative of $f(x) = 3^x$. $\frac{d}{dx}\left(3^x\right) =$ ____________________ Try it!

What is the derivative of $f(x) = \left(\frac{1}{2}\right)^x$? $\frac{d}{dx}\left(\left(\frac{1}{2}\right)^x\right) =$ ____ Try it

So, in general:

The Exponential Rule

For any positive constant a, $\frac{d}{dx}\left(a^x\right) =$ ______________________________

And again, as a special case of the Exponential Rule,

$\frac{d}{dx}\left(e^x\right) =$ ____

Examples:

1. If $f(x) = \frac{5^x}{3} - 9e^x$, find $f'(x)$.

2. Find the <u>equation of the tangent line</u> to the graph of $f(x) = 4\left(\frac{1}{3}\right)^x$ at $x = 0$. Check your answer by graphing f and the tangent line.

3. Once again, the **Cliff Notes** problem.

We found the function that solved the Cliff Notes problem was $P(t) = 4000(1.226)^t$, where t represented the number of years after 1958 and P was the value of the company in that year. Find P'(12) and interpret your answer.

The Derivative of the Natural Logarithmic Function (Section 3.2) **Date:** ______________

Before we begin our new topic of the "derivative of the natural logarithmic function", let's look at one more application problem of exponential decay.

Radioactive Decay: Radioactive substances decay by spontaneously emitting radiation. These substances decay at a rate proportional to to the mass of the substances. This means that the mass of the substance can be modeled by our exponential decay functions; either the <u>continuous growth model</u> $P(t) = P_0 e^{kt}$ or the <u>discrete growth model</u> $P(t) = P_0 a^t$. Physicists express the rate of decay in terms of **half-life,** the time required for half of any quantity to decay.

Example: The radioactive substance Bismuth-210 has a half-life of 5.0 days. A sample originally has a mass of 800 mg.

a. Find the <u>continuous growth formula</u> for the mass remaining after t days.

P(t) = ______________

b. Convert the <u>continuous growth model</u> to the corresponding <u>discrete growth model</u>. What is the "daily decay rate"?

P(t) = ______________

Decay rate: __________

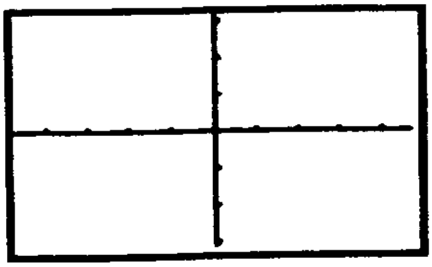
d. Use the <u>discrete growth model</u> (part b), and find the rate of change of the sample on the 5th day. On the 15th day.

Day 5: ____________

As we have done with many of our derivative properties, we will attempt a graphical approach in discovering the derivative of the natural logarithmic function, $f(x) = \ln x$.

Sketch the graph of $f(x) = \ln x$ below. (Can you do it without your calculator?)

f

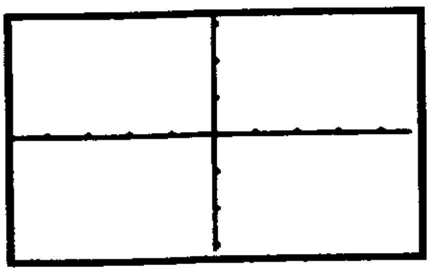


Label a few points that you know are on the graph of f.

By looking at the graph above, what information do you know about the derivative of the natural log function?

Use this information and sketch $f'(x)$, the derivative of $f(x) = \ln x$.

f'



Are we able to make a guess as to the function that we just graphed? Hint: It's a function that we know!

Conjecture: If $f(x) = \ln x$, then $f'(x) =$ ________ Or, using Leibniz notation ____________________

Let's check this conjecture graphically.

Y1 = ln(x)   Y2 = nDeriv(Y1 , X , X)   Y3 = (Our guess)

Before we do any examples of this new derivative property, let's write a summary of the derivative properties we now know.

Assume f and g are functions, and a and k are constants.

1. $\frac{d}{dx}(k) =$ ______   2. $\frac{d}{dx}(x) =$ ________   3. $\frac{d}{dx}(k \cdot x) =$ ____________

4. $\frac{d}{dx}(x^n) =$ ________   5. $\frac{d}{dx}(f(x) \pm g(x)) =$ ____________   6. $\frac{d}{dx}(k \cdot g(x)) =$ ____________

7. $\frac{d}{dx}(e^x) =$ ______   8. $\frac{d}{dx}(a^x) =$ ________   9. $\frac{d}{dx}(\ln x) =$ ____________

Examples:

1. If $f(x) = \frac{A}{x^5} + Be^x - C\ln x$, find f'(x).

Note: Don't forget that the point of all of the derivative formulas is to use them to solve problems!

2. Let f(x) = ln x.

   a. Find f'(1/2) , f'(2), and f'(10).

   f'(1/2) = ____________ f'(2) = ________ f'(10) = ________

   When you compare these values, what do they tell you about the graph of f?

   b. Find f''(2). Does this answer confirm your answer above?

3. **Web TV.** The number of homes with access to the Internet by way of cable television between 1998 and 2005 can be modeled by the equation:

   I(x) = -138.27 + 76.29 ln x, where I(x) is measured in million homes x years after 1990.

   a. Find I(12) and interpret your answer (with units).

   b. Find I'(12) and interpret your answer (with units).

4. If $1000 is invested in a bank at an annual interest rate of 4.5%, and the interest is compounded continuously, then the amount of money M(t) in the account after t years is given by the equation

$$M(t) = 1000e^{0.045t}$$

a. Solve this equation for t in terms of M; i.e. find t(M).

b. Use the answer to part a to determine how long it will take for the investment to triple in value.

c. Find $\left.\frac{dt}{dM}\right|_{M=1000}$ and interpret your answer (with units).

$$\left.\frac{dt}{dM}\right|_{M=1000} =$$

A New and Important Derivative Property - The Chain Rule (Section 3.3) Date: ____________

I. Review of Composition of Functions

1. If $f(x)=|x|$, $g(x)=1-x^2$, and h(x)=ln x, what is:

   a. f(g(5))= ____________________

   b. $g\left(h\left(e^3\right)\right)=$ ____________________

   c. f(g(x))= ____________________

   d. g(h(x))= ____________________

   e. h(g(x))= ____________________

2. Look at this problem in reverse.

   a. If h(x)=f(g(x)) and $h(x)=2e^{5x+1}$,

      what is f(x)= ____________________

      g(x)= ____________________

   b. If h(x)=f(g(x)) and $h(x)=\left(3x^2+1\right)^3$,

      what is f(x)= ____________________

      g(x)= ____________________

   c. If h(x)=f(g(x)) and $h(x)=\sqrt{\ln x}$,

      what is f(x)= ____________________

      g(x)= ____________________

II. Apply composition of functions to our calculus

1. Make a guess.

   a. We know: If $y = 2e^x$, then y'= ______________________________

      Guess: If $y = 2e^{5x+1}$, then y'= ______________________________

   b. We know: If $y = x^3$, then y'= ______________________________

      Guess: If $y = (3x^2+1)^3$, then y'= ______________________________

   c. We know: If $y = \sqrt{x}$, then y'= ______________________________

      Guess: If $y = \sqrt{\ln x}$, then y'= ______________________________

2. Checking the guesses.

   a. Check the guess for Part a above <u>graphically</u>.

      Let: Y1= $2e^{5x+1}$

      Y2= ______________________ (The guess)

      Y3= nDeriv(Y1,X,X)

      Deactivate Y1 and graph Y2 and Y3.

      Was the guess correct? __________

      The correct answer is: If $y=2e^{5x+1}$ then y'= ______________________________

   b. Check the guess for Part b above algebraically.

      If $y = (3x^2+1)^3$, expand the right side of the equation, then find the derivative.

$$\begin{aligned} y &= (3x^2+1)^3 \\ &= (3x^2+1)(3x^2+1)(3x^2+1) \\ &= (3x^2+1)(9x^4+6x^2+1) \\ &= 27x^6+27x^4+9x^2+1 \end{aligned}$$

      y' =

c. If we see the pattern, correct the guess for Part c, and check the answer graphically.

If $y = \sqrt{\ln x}$, then y'=______________________________

Let: $Y1 = \sqrt{\ln x}$

Y2= __________________________ (New guess for y')

Y3= nDeriv(Y1,X,X)

Deactivate Y1 and graph Y2 and Y3.

III. Conclusion and Generalizations

1. We saw that if $h(x) = 2e^{5x+1}$, then $h'(x) = 2e^{5x+1} \cdot 5 = 10e^{5x+1}$.

Therefore, if $y = e^{g(x)}$, then y' = __________________________________

2. We saw that if $h(x) = \left(3x^2+1\right)^3$, then $h'(x) = 3\left(3x^2+1\right)^2 \cdot 6x = 18x\left(3x^2+1\right)^2$

Therefore, if $y = (g(x))^n$, then y' = ________________________________

3. This property for finding derivatives of a "composition of functions" is called the **Chain Rule.**

One more time, in the general form, the **Chain Rule** says:

If y = f(g(x)), then y'= ____

Using Leibniz notation,

If y = f(g(x)), let z = g(x). Then y = f(z), and $\frac{dy}{dx} =$ ____________________________

IV. Examples:

1. Find derivatives of the following functions

a. $s = (4t-1)^5$

b. $y = \sqrt{4-9x^2}$

c. $f(x) = (\ln x)^4$

d. $y = \sqrt[3]{(e^x + x)^2}$

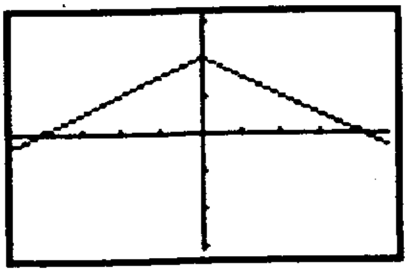
2. Suppose $5000 is deposited in a bank account that pays 4.5% annual interest, compounded continuously.
   a. Find a formula for P(t), the balance in the account t years after the initial deposit.

      P(t) = ______________________________

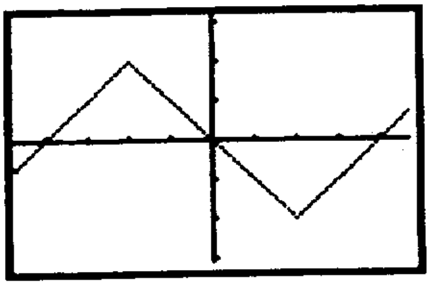
   b. Find P(10) and P'(10) and interpret your answers with units.

3. Use the figures below, to evaluate the expressions.

f(x)



g(x)



a. g(3) = ______________________

b. f(-3) = ______________________

c. f(g(2)) = ______________________

d. g(f(1)) = ______________________

e. $\left.\frac{d}{dx}f(x)\right|_{x=-2}$ = ______________________

f. $\left.\frac{d}{dx}g(x)\right|_{x=4}$ = ______________________

g. $\left.\frac{d}{dx}f(g(x))\right|_{x=1}$ = ______________________

h. $\left.\frac{d}{dx}g(f(x))\right|_{x=-1}$ = ______________________

Two New Derivative Properties (**Section 3.4**) **Date:** ____________
The Product Rule and the Quotient Rule

I. The Product Rule

Introduction: We already have a property that allows us to find the derivative of a **sum** (or difference) of two functions.

Example:
If $f(x) = x^3 + x^5$,
then $f'(x) =$ ________ + ________

Property:
If $f(x) = g(x) + h(x)$,
then $f'(x) =$ ________ + ________

Objective: We want to discover a property that can be applied to find the **product** of two functions.

Let $g(x) = x^3$ and $h(x) = x^5$. Now, let $f(x) = g(x) \cdot h(x) =$ ________ • ________ = ________.

Find: $f'(x) =$ ________ $g'(x) =$ ________

$h'(x) =$ ________

$g'(x) \cdot h'(x) =$ ________

So, notice that although it is true that : **If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$,**

it is not true that: **If $f(x) = g(x) \cdot h(x)$, then $f'(x) = g'(x) \cdot h'(x)$.**

It is possible to get the correct answer for $f'(x)$ by a clever combination of the equations for $g(x)$, $h(x)$, $g'(x)$, and $h'(x)$.

Write the functions again.

$g(x) =$ ______, $h(x) =$ ______, $f(x) =$ ______, $g'(x) =$ ______, $h'(x) =$ ______, $f'(x) =$ ______
(Remember, we are trying to get the function $f'(x)$ from a combination of $g(x)$, $h(x)$, $g'(x)$, and $h'(x)$.)

See if you can figure out what this combination is (with the following hints)!

Notice that the **8** in $f'(x) = 8x^7$ is the sum of the **3** and **5** in $g'(x) = 3x^2$ and $h'(x) = 5x^4$.

Fill in the following

$$f'(x) = 8x^7 = 5x^7 + 3x^7 = \_\_\_\_\_\_\_\_ \cdot 5x^4 + \_\_\_\_\_\_\_\_ \cdot 3x^2.$$

Notice what the functions are that you put in the blanks!

Now try to complete the conjecture. This derivative property is called the **Product Rule.**

Product Rule: If $f(x) = g(x) \cdot h(x)$, then $f'(x) =$ ________ · ________ + ________ · ________

When you think you have it, try the following. Assume that your conjecture is true for the product of any two functions.

If f(x) = $f(x) = x^2 \cdot \ln x$, then $f'(x)$ = __________ · __________ + __________ · __________.

= ______________________________

or ______________________________

Check your result graphically by graphing f(x) in Y1, your derivative in Y2, and the nDeriv(Y1,X,X) in Y3.

Try one more.

If $f(x) = x^3 e^{4x}$, then $f'(x) =$ __________ · __________ + __________ · __________.

= ______________________________

or ______________________________

Again, you can check your result graphically.

Examples of the Product Rule:

1. Let $f(x) = (3x^2 + 4)(2x^2 + 3)$

   a. Find $f'(x)$ by using the Product Rule.

   b. Find $f'(x)$ by expanding the terms first, then applying the Power Rule.

   c. Show that the two answers are equivalent.

2. Let $f(x) = x^2(x+3)^2$

a. Graph f on your calculator.

b. From the graph of f, for what x-values does it appear that f has a horizontal tangent?

x= ______________________________

c. Analytically, find all values of x where the graph of f has a horizontal tangent line.

3. If $f(x) = x \cdot e^x$, find f'(x), f''(x), f'''(x), and a rule for $f^{(n)}(x)$, the nth derivative of f(x).

f'(x) = ______________________________

f''(x) = ______________________________

f'''(x) $= f^{(3)}(x) =$ ______________________________

$f^{(n)}(x)$ = ______________________________

Note: So, the above example shows that it's necessary to use the Product Rule to find the derivative of $f(x) = x \cdot e^x$. Is it necessary to use the Product Rule to find the derivative of $f(x) = 8 \cdot e^x$? Why or why not?

Could you use the Product Rule to find the derivative of $f(x) = 8 \cdot e^x$?

II. The Quotient Rule

New Objective: We want to discover a property that can be applied to find the **quotient** of two functions.

Actually, in many instances, the property is not necessary. For example, if we wanted to find the derivative, f'(x). for $f(x)=\frac{2^x}{x^3}$, we could rewrite the function as $f(x)=\frac{2^x}{x^3}=$ ____________________ and then find the derivative using the Product Rule. Lets do this!

f'(x) =

However, we will see that most of the time it is easier to leave the function as a quotient, and find the derivative without rewriting it.

Let's see if we can discover this property by using the preceding example. Follow the steps below, look at the fi answer, and see if you can see a pattern!

Let $f(x)=\frac{2^x}{x^3}$.

So, if $f(x)=\frac{2^x}{x^3}$, then $f'(x)=$ ______________________________

(Note: Is this answer the same as our answer when we did the problem by rewriting the function and using the Product Rule?)

The result of this problem is the property called the **Quotient Rule**. The **Quotient Rule** says that:

> If $f(x)=\frac{g(x)}{h(x)}$, then $f'(x)=$ ______________________________

It's time to practice!

Examples of the Quotient Rule:

1. If $f(x)=\frac{5x^2}{3x+5}$, find $f'(x)$.

2. Let $f(x) = \frac{x}{x^2+1}$.

a. Graph the function on your calculator and determine intervals where the function is increasing and decreasing.

Increasing ______________________ Decreasing ____________________________

b. To verify the intervals above, find:

$f'(-2) =$ _____ $f'(-1) =$ _____ $f'(0) =$ _____ $f'(1) =$ _____ $f'(2) =$ _____

3. **Practice Problem**

Show that the derivative of $f(x) = \frac{ax+b}{cx+d}$ is $f'(x) = \frac{ad-bc}{(cx+d)^2}$, where a, b, c, and d are constants.

4. Write the equation of the tangent line to the function $y = \frac{e^x}{e^x+1}$ at $x = 0$. Verify your answer graphically.

5. Given $f(x)=\dfrac{k}{g(x)}$. Find the derivative $f'(x)$ two different ways.

a. By using algebra to rewrite the function (using a negative exponent), then differentiating.

$$f(x)=\frac{k}{g(x)}=\underline{\hspace{6cm}}$$

b. By using the Quotient Rule.

6. Given $f(x)=\ln\left(\dfrac{x-1}{x+1}\right)$. Find the derivative $f'(x)$ two different ways.

a. By applying calculus right away, then simplifying the answer.

b. By simplifying the expression first, then applying calculus.

$$f(x)=\ln\left(\frac{x-1}{x+1}\right)=\underline{\hspace{16cm}}$$

c. Show that the answers for parts a and b above are equivalent.

Critical Points, Local Maxima, and Local Minima **(Section 4.1)** **Date:** ____________

An Old Question (to get us started). What does the derivative of a function tell us about the function and the graph of the function?

Introductory Example: On your calculator, use a "Standard Window" (ZOOM, 6:ZStandard) and graph the function:

$$f(x) = x^3 - 9x^2 - 48x + 52 .$$

If we are interested in knowing on what intervals the function is increasing and on what intervals it is decreasing, obviously this graph does not help!

So, we could "play" with the window of our calculator to get a better picture of the function, or we can solve the problem analytically!

First, we need to find f'(x). f'(x) = ______________________________

Second, before we find where f'(x) > 0 or f'(x) < 0, let's find where ____________. (These numbers are called **critical numbers,** or **critical points**, of the function.)

So, the critical numbers of f(x) are x = ________ and x = ________.

Now, how can we find the intervals where the function is increasing/decreasing?

1. Let's make a chart like a number line. Label the information we just found.

2. Pick a number on the number line between each critical number and substitute it into f'. Remember, we only care if this value is positive or negative. Label this information on the chart.

Let's take this problem one step further. Can we tell where the maximum and minimum points on the graph are? How?

3. From the chart (number line) above, find the maximum and minimum points on the graph of the function. (These are called **local maxima** and **local minima** points of f.)

   Maxima: ____________________ Minima: ____________________

Now we see why not much showed up in our first calculator graph!

4. What's the y-intercept of the graph? ____________

5. Use all of the above information, and sletch a graph of the function. Label the points that we found.

6. Adjust the plotting window on your calculator and sketch the graph again to verify our work.

Practice Problem: Given $f(x) = x^4 - 4x^3$, find the critical points, intervals where the function increases and decreases, and local maxima and minima points.

Critical points: ______________ Increasing: ______________ Decreasing: ____________________

Local Maxima: ______________ Local Minima: ______________

Let's summarize what we have learned from the examples:

1. **Definition**: A point (p , f(p)) is a **critical point** of a function f if ______________________

2. **Property** (called the **First Derivative Test**): Suppose p is a critical number of a function f.

   - If f changes from decreasing to increasing at p, then f has a ______________________

   - If f changes from increasing to decreasing at p, then f has a ______________________

The second derivative can also tell us whether or not there is a local maximum or minimum at a critical number p.

Recall what the second derivative tells us about a function f. Where f''> 0, f is ______________________

and where f''< 0, f is ______________________.

3. **Property** (called the **Second Derivative Test**): Suppose p is a critical number of a function f, and f'(p) = 0.

   - If f is concave up at p, then f has a ______________________ at x = p.

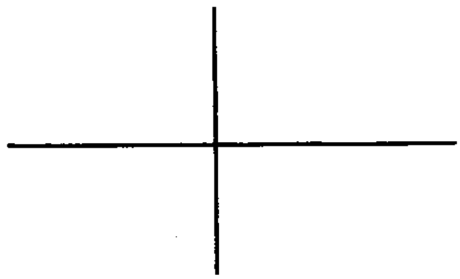
   - If f is concave down at p, then f has a ______________________ at x = p.

Example: Use the **Second Derivative Test** to verify that $f(x) = x^3 - 9x^2 - 48x + 52$ has a local maximum at x = -2 and a local minimum at x = 8, that we found in an earlier example.

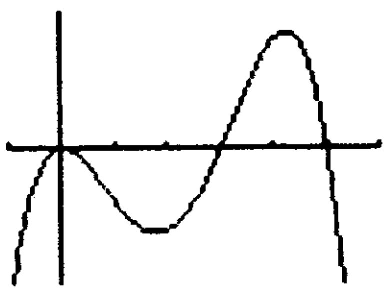
Note: Although it is a true statement that "If a function f has a local maximum or a local minimum at a number p, then p is critical number of f", it is not true that " If a function f has a critical number at a point p, then f is a local maximum or local minimum of f". Can you think of a simple function that verifies this?

Examples:

1. Graph a function with 3 critical points, one a local maximum, one a local minimum, and the other neither a local maximum nor a local minimum.



2. Below is a graph of f', the derivative of a function f. What are the critical points of the function? Over what intervals is the function f increasing and decreasing? For what values of x does f have a local maximum and a local minimum?



Critical numbers: ______________________

Increasing: ______________________

Decreasing: ______________________

Local Maximum: ______________________

Local Minimum: ______________________

Concavity and Inflection Points (Section 4.2) **Date:** ______________

Last class period we saw that a **critical point** was a point on the graph where the slope (f') changes sign. Now we will study the points where the concavity (f'') changes, either from concave up to concave down, or concave down to concave up.

Definition: A point at which the graph of a function f changes concavity is called an **inflection point**.

How do we locate an inflection point? Since the concavity of the graph of f changes at an inflection point, the sign of f'' changes there.

Therefore, at an inflection point, ____

Examples:

1. Find inflection points for the following functions.

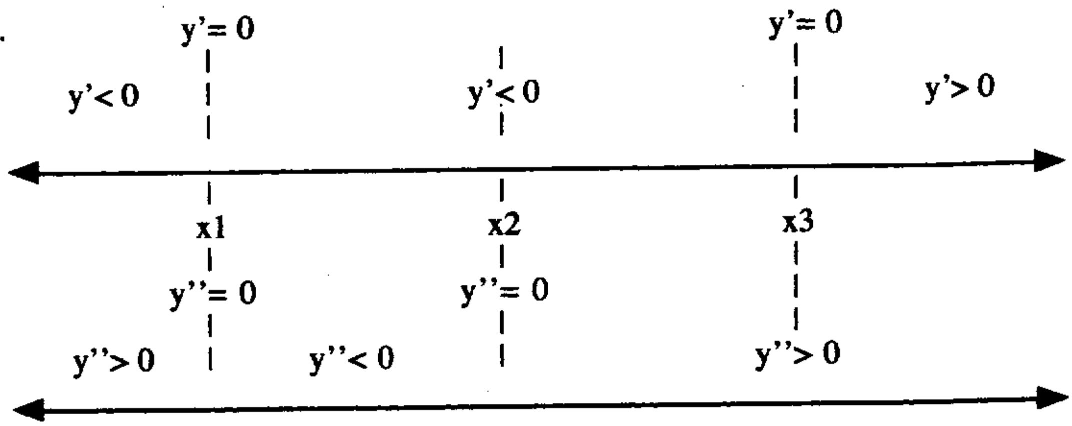
a. $f(x) = x^4 - 4x^3$

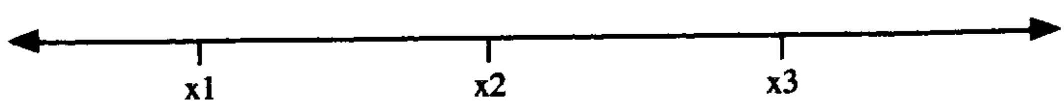
b. $g(x) = \ln(1 + x^2)$

2. Find the critical points and inflection points of $f(x) = x \cdot e^{-x}$. (This is a function we will study closely in the near future for the Pharmacy majors.)

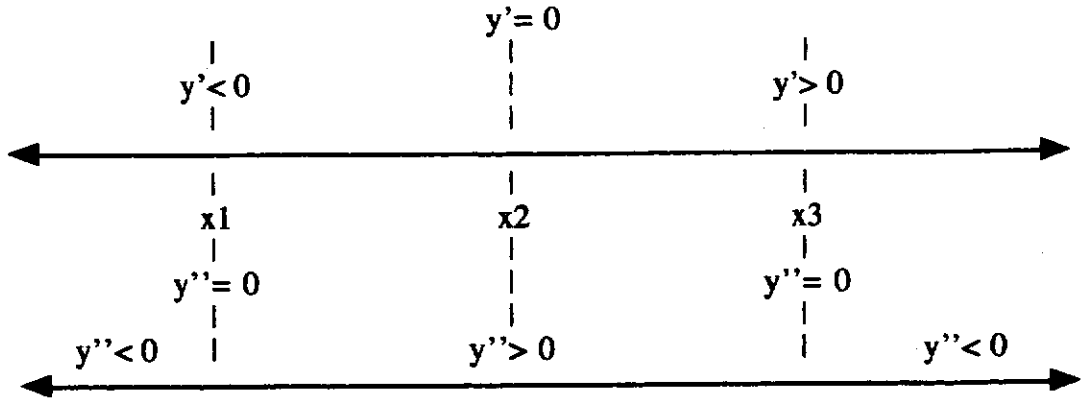
3. Using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$, sketch a possible graph of $y = f(x)$. Assume that the function is defined and continuous for all x.

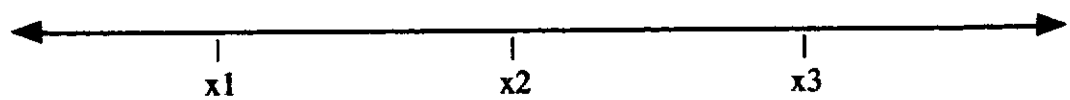
a.





b. **Practice Problem:**





Graphing Functions - Putting It All Together **(Section 4.2 cont.)** **Date:** ___________

Below you are given a function f, its first derivative f', and its second derivative f''.

1. Find all **intercepts** of the function.
2. Find the **critical points** of the function.
3. State intervals where the function is **increasing** and **decreasing**.
4. Find all **local maximum** and **local minimum** points.
5. Find the **possible inflection points** of the function.
6. State intervals where the function is **concave up** and **concave down**.
7. Find all **inflection** points.
8. Using the above information, **sketch the graph** of f. Label the intercepts, maximum, minimum, and inflection points.
9. You may want to check your answer by graphing the function on your calculator.

Example: $f(x) = x(x-4)^3$, $f'(x) = 4(x-1)(x-4)^2$, $f''(x) = 12(x-2)(x-4)$

Practice Problem: $f(x) = x(2x+9)^2$, $f'(x) = 3(2x+3)(2x+9)$, $f''(x) = 24(x+3)$

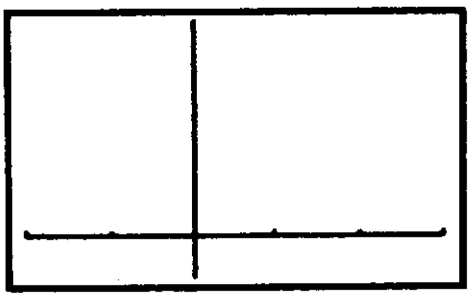
Global Maxima and Minima **(Section 4.3)** **Date:** ____________

Let's begin with an example.

Example: 1. Find all **local minima** and **local maxima** for the function $f(x) = x^4 - 2x^2 + 3$.

Local minimum: ______________ Local maximum: ____________________

2. On your calculator, sketch the graph of f on the interval [-2 , 3]. Show the graph below. Label the local maximum and local minimum.



3. We are often interested in where a function is larger or smaller than <u>all other points</u> on an interval. These points are called **global** (or absolute) **maxima** and **minima**. On the interval [-2 , 3], what are the **global minimum** and **global maximum** values of f?

Global minimum: ______________ Global maximum: ____________________

So, to summarize: **How do we find global maxima and global minima on a closed interval?**
(Note: A closed interval is an interval that includes the endpoints.)

Note: Given a continuous function defined on a closed interval [a , b], the function <u>must have</u> a global maximum and global minimum somewhere in [a , b]. (This called the **Extreme Value Theorem.**)

But what if we are interested in finding global maxima and global minima on an interval that is not a closed interval.; i.e., an interval that does not contain an endpoint. Or an interval that includes ∞.
Let's discuss this through an example.

Example: Consider the function $f(x)=\dfrac{\ln x}{x}$. The derivative is $f'(x)=\dfrac{1-\ln x}{x^2}$. (You should verify this!)

1. With your calculator, sketch f.

2. Find the critical number(s).

The critical number(s) is x = __________, and the critical point is (________ , ________)

3. State the **global maxima** and **global minima** (if they exist) on the interval indicated.

a. [1 , ∞] Max: ________ Min: __________

b. (0 , e] Max: ________ Min: __________

c. [e , ∞] Max: ________ Min: __________

d. (0 , ∞] Max: ________ Min: __________

Examples:

Graph a function with the given properties.

a. Has a local maximum at $x = -2$, a local minimum at $x = 3$, but no global maximum or minimum.

b. Has no local or global maxima or minima.

c. Has a local maximum and global maximum at $x = -2$ but no local or global minimum.

d. Has a local and global maximum at $x=-1$, and a local and global minimum at $x=3$.

Modeling and Optimization Problems **(Section 4.3 cont.)** **Date: __________**

From the previous few sections, we see that it is important to be able to find a local max/min or a global max/min of a function. But, when solving problems in the real world, it is very seldom that we are given the specific function to work with. Therefore, we need to build the function to which we will then apply our calculus concepts. These examples and problems are mainly "geometric" in nature, rather than problems from your area of study.

Examples:

1. Twenty-five feet of fence is to be put around a garden. The plans have one edge of the garden to be along the side of a house, with the fence enclosing the other three sides of a rectangle. Assume that the width of the garden is the side perpendicular to the house, and the length of the garden is the side parallel to the house,

a. Draw a figure that illustrates the problem.

b. If the width of the garden is 3 feet, what is the length of the garden? What is the area?

Width = _________ Length = _________ Area = __________

c. If the width of the garden is 10 feet, what is the length of the garden? What is the area?

Width = _________ Length = _________ Area = __________

d. If the width of the garden is x feet, write a function A(x) which represents the area of the garden as a function of the width.

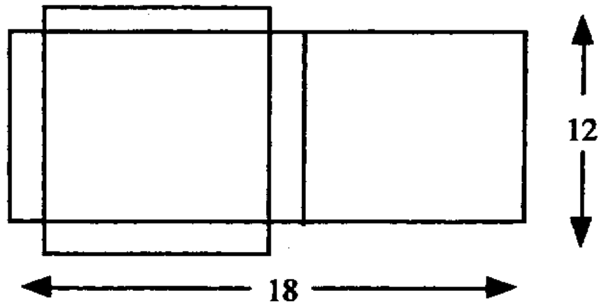
A(x) = ____

e. Analytically, find the exact width of the garden that will maximize the area. What is the length? What is the maximum area?

Width: ___________ Length: ___________ Area: ___________

f. Check your answer graphically

2. A box with a top is to be constructed out of a 12 inch by 18 inch piece of cardboard by cutting congruent squares from two of the four corners and rectangles from the other two corners, then folding up the sides (as shown below).



a. If the square in the upper left (and lower left) hand corners is **2 inches**, what is the length of the box? ____________ What is the width of the box? ______________ What is the height of the box? ______________ What is the volume of the box? ________________

b. If the square in the upper left (and lower left) hand corners is **x inches**, write an expression for the length of the box? ______________________ Write an expression for the width of the box? __________________ Write an expression for the height of the box? __________

Write a function V(x) for the volume of the box?

V(x) = ____

c. Analytically, determine how large the cut-out squares should be to maximize the volume of the box. What is the maximum volume? Check your answer graphically.

3. You are planning to make an open rectangular box with a square base that will hold a volume of 50 cubic feet.

a. Sketch a picture of the problem.

b. If a side of the base is 2 ft, what is the height of the box? _______________ What is the surface area of the box? _______________

c. If a side of the base is 5 ft, what is the height of the box? _______________ What is the surface area of the box? _______________

d. If the side of the base is x ft, find an algebraic representation S(x) for the surface area of the box.

S(x) = _______________

e. Using the function S(x), analytically determine the dimensions of the box if the surface area is to be as small as possible; that is, minimize the amount of material being used. What is the minimum amount of material used?

Minimum width: __________ Minimum length: __________

Minimum height: __________ Minimum surface area: __________

Problems for Assignment: (Do on another sheet of paper!)

4. A farmer with 5,000 feet of fencing wants to enclose a rectangular field and then divide it into two plots by adding a fence in the middle parallel to one of the sides.

   a. Sketch a picture of the problem.

   b. What is the area of the field if the width (one of the sides parallel to the added fence) is 500 ft?

   c. What is the area of the field if the width is 1200 ft?

   d. If the width of the field is x ft, find an algebraic representation A(x) for the area of the field.

   e. Using the function A(x), analytically find the dimensions of the field that would maximize its area.

5. A rectangular storage container with an open top is to have a volume of 32 ft^3. The length of the base is equal to the width. Material for the base costs \$6 per square foot and material for the sides cost \$4 per square foot.

   a. Sketch a picture of the problem.

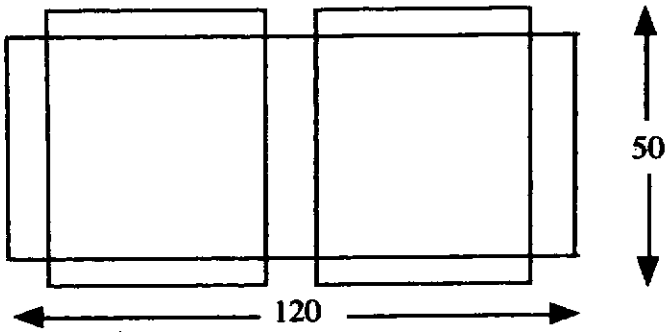
   b. If the width (and length) of the base is 2 feet, what is the height of the container? What is the area of the base? What is the cost of the material to make the base? What is the area of one side? What is the cost of the material for the 4 sides? What is the cost of the material for the entire container?

   c. If the width (and length) of the base is 3 feet, what is the height of the container? What is the area of the base? What is the cost of the material to make the base? What is the area of one side? What is the cost of the material for the 4 sides? What is the cost of the material for the entire container?

   d. If the width (and length) of the base is x feet, write an algebraic representation for the height of the container. The area of the base. The cost of the material to make the base. The area of one side of the container. The cost of the material for the 4 sides. Finally, write a function C(x), that determines the total cost of the material for the entire container.

   e. Using the function C(x), analytically find the dimensions of the container, and the cost of the materials for the cheapest such container.

6. A pizza box is to be constructed out of a 50 by 120 cm piece of cardboard by cutting six equal-sized squares in the positions shown, then folding up the sides (as shown below).



a. If the squares cut out of the six places of the cardboard measure **7 cm**, what is the length of the pizza box? What is the width of the pizza box? What is the height of the pizza box? What is the volume of the pizza box?

b. If the squares cut out of the six places of the cardboard measure **20 cm**, what is the length of the pizza box? What is the width of the pizza box? What is the height of the pizza box? What is the volume of the pizza box?

c. If the squares cut out of the six places of the cardboard measure **x cm**,

- Write an expression for the length of the pizza box?

- Write an expression for the width of the pizza box?

- Write an expression for the height of the pizza box?

- Write a function V(x) for the volume of the pizza box?

d. Find how large the cut-out squares should be to maximize the volume of the pizza box. What is the maximum volume? What is the width of this box? What is the length of this box?

A New Function - The Logistic Growth Function (Section 4.7) **Date: ____________**

We will introduce this new function through an interesting example!

The STARBUCKS Problem:

The first Starbucks store opened in 1971 at Pike Place Market in Seattle. The chart below shows the growth of the company from the year 1988.

Year	1988	1990	1992	1994	1996	1998	2000	2002
Number	33	84	165	425	1015	1886	3501	5886

1. Create a scatterplot of the data, with the year representing number of years from 1988; i.e. enter 1988 as 0, 1990 as 2, etc.

2. What type of function would seem to best model the growth of Starbucks? ________________

3. Using the model $n(t) = a \cdot b^t$, analytically find a function that fits the data. (You need to find values for "a" and "b".) On your calculator, graph n(t) to see how it fits the scatterplot.

n(t) = ________________

4. Use **ExpReg** (Exponential Regression) to find the exponential function n(t) that best fits the data. (From the Home Screen, press STAT, right arrow to CALC, Enter, 0:ExpReg.) On your calculator, graph n(t).

n(t)= ________________

5. Using the exponential regression function, find the projected number of Starbucks in the year 2004? (Note: You may want to turn off the scatterplot and the function found in #3 above.)

n(_____) = ________________

6. Using the exponential regression function, find when the number of Starbucks will reach 11,000. Solve this problem algebraically, then check your answer graphically.

t = ________________

7. Remember, in addition to writing an exponential function in the form $n(t) = a \cdot b^t$, we can also write the exponential function in the form $n(t) = a \cdot e^{kt}$, but we need to find the value of k. Find the value of k and rewrite the function in this form.

k≈ ____________ and n(t) = ______________________

8. Above, we determined that, according to the exponential regression model, the projected number of Starbucks in the year 2004 would be ____________. We now know that in 2003 there were 7225 Starbucks, and in 2004 there were 8337 stores. Add these values to the scatterplot. How does the exponential growth model fit the function with these new values?

9. A better fitting model now would be a **logistic function**. Using the regression capabilities of the calculator, find a logistic function (B:Logistic) that fits the data.

n(t) = ________________________________

10. Does the logistic growth model show an "upper limit" to the number of Starbucks stores? Change the window to see if the graph appears to have a horizontal aymptote as x gets bigger. If it does, what is it?

y = _______________

Where does this number appear in the logistic function? ____

11. Use the logistic function to estimate when the number of Starbucks reaches 11,000. Solve the equation graphically.

t = __________

12. Find n'(10), and interpret its meaning with units. How does this answer compare to the information from the chart on the preceding page?

The Logistic Function

A logistic function, such as that used to model the growth of Starbucks, is everywhere positive ($f(x) > 0$), and everywhere increasing, ($f'(x) > 0$). Its graph is concave up at first ($f''(x) > 0$), then becomes concave down ($f''(x) < 0$), and levels off at a horizontal asymptote.

Also, a logistic function is approximately exponential for small values of t.

Logistic functions can also be used to model the growth of a population in a closed environment, the sales of a new product, and the spread of a virus or disease.

For positive constants L, C, and k, a logistic function has the form:

$$P = f(t) = \frac{L}{1 + Ce^{-kt}}$$

Notice that the general logistic function $P = f(t) = \frac{L}{1 + Ce^{-kt}}$ has three parameters: L, C, and k.

Let's investigate the effect that two of these parameters (L and k) have on the graph of a logistic function.

Example: Consider the logistic function $P = \frac{L}{1 + 100e^{-kt}}$.

a. Let $k = 1$. Graph P with $L = 1$, $L = 2$, and $L = 4$, to see the effect of L on the function.

The value of L determines the ____

Note: This value is called the **carrying capacity.**

b. Now let $L = 1$. Graph P with $k = 1$, $k = 2$, $k = 3$, to see the effect of k on the function.

The value of k determines the ____

It is often important (but difficult) to predict the carrying capacity of a logistic model. For example, a manufacturing company may want to estimate the maximum potential sales of a new product.

One way of estimating the carrying capacity is to find the inflection point. At the inflection point, where the concavity changes from concave up to concave down, the slope of the curve is the largest. (For this reason, the inflection point in an economics application is called the <u>point of diminishing returns</u>.)

It can be shown, of course using calculus, but not easily, that the inflection point occurs where the y-value $P = \frac{L}{2}$.

Note: For the logistic function that models the growth of the Starbucks stores, $n(t) = \frac{12841}{1 + 307e^{-.397t}}$, the inflection point would occur when y = ____________. What year did this occur? ____________

Example: The table below shows the total sales (in thousands) of a new CD since it was introduced.

t (in months)	0	1	2	3	4	5	6	7
P (total sales in thousands)	0.5	2	8	33	95	258	403	496

1. After how many months does the function that models the sales of the CD change concavity? ______

2. Use this value to estimate the maximum potential sales, L. ______________________

3. Using a logistic regression, find the best fitting logistic function to this data.

   P(t) = ______________________

4. What maximum potential sales does the regression equation predict? ______________________

5. According to the regression equation, at what time does the point of diminishing returns occur? (Note: Watch how this is found!)

Another New Function - The Surge Function and Drug Concentration **(Section 4.8)** **Date:** ____________

Again, we will introduce this new function through an example.

The concentration of bemetizide (a diuretic) in the blood after a single oral dose of 25 mg is given below.

t (hours)	0	1	2	3	4	6	8	12	16
concentration (ng/ml)	0	22	57	80	82	78	60	36	25

Create a scatterplot of the data.

Functions with behavior such as this are called **surge functions.** They have equations in the form:

$y = ate^{-bt}$, where a and b are positive constants.

Let's look at the "family of curves" in this form, and see what effect the parameters "a" and "b" have on the shape of the graph.

Let $a = 1$. Graph y with $b = 1$, $b = 2$, and $b = 3$, to see the effect of b on the function.

Notice that the general shape of the curve does not change as b changes, but as b increases, the curve rises for a shorter period of time and has a smaller maximum value.

Where does the maximum occur for each of the three values of b.

When $b = 1$, $t =$ ________ , and when $b = 2$, $t =$ ________ , and when $b = 3$, $t =$ ________

It appears that the maximum value of $y = te^{-bt}$ ocurs at $t =$ ________. Let's prove this!

And, if we want to find the maximum value of the function, we can substitute 1/b for t in $y = te^{-bt}$ and evaluate y.

Conclusion: The function $y = te^{-bt}$ has a maximum at the point (________ , ________)

Now, let's look at the effect that the parameter "a" has on the shape of the graph in the function $y = ate^{-bt}$

Let b = 1. Graph y with a = 1, a = 2, and a = 3, to see the effect of a on the function.

Does it effect the x-value of the maximum point? __________ (We could prove this!)

Does it effect the y-value of the maximum? ___________ The y-value is now y =____________.

Conclusion: The function $y = ate^{-bt}$ has a maximum at the point (________ , ________)

This result can allow us find values of "a" and "b" in the surge function $y = ate^{-bt}$, if we know where the maximum point is. And, this would allow us to find an approximate model for the data that we have. Let's see how this works!

Since we know the maximum occurs at t = 1/b, we can solve this equation for b. b = ________

And the y-value of the maximum point occurs when y = _________. Solve this equation for a. a = __________

Let's see how this works for our "drug concentration" example. Here is the table of values again.

t (hours)	0	1	2	3	4	6	8	12	16
concentration (ng/ml)	0	22	57	80	82	78	60	36	25

The maximum value of ____________ occurs when t = _________. Use this to solve for a and b, and write the surge function that (hopefully) approximates the data.

b = _____________ , a = ______________________________ , y = ______________________________

Graph it to see how it fits!

Minimum Effective Concentration: The minimum effective concentration of a drug is the blood concentration necessary to achieve a pharmacological response. The time at which this concentration is reached is referred to as **onset**, and the time that the drug concentration falls below this level, is when **termination** occurs.

Back to the example! Suppose the minimum effective concentration of bemetizide in a patient is 40 ng/ml. Find the time until the onset of effectiveness, the termination of effectiveness, and the total time that the drug is effective.

Onset: t =____________ Termination: t = _____________ Total time of effectiveness = ____________

Introduction to the Definite Integral (Section 5.1) Date ____________
Distance and Accumulated Change

So, what have we studied this far into the course? We began by looking at the "rate of change" of a function. The average rate of change was calculated as the slope of a secant line. Then we estimated an instantaneous rate of change, the slope of the tangent line, by calculating the average rate of change over a small interval. This led to the definition of the derivative function, which tells us the rate of change of a function at any value. Finally, we have looked at applications of the derivative which shows how the derivative can give us more information than just a slope of a tangent line.

Now, we can consider the reverse process. Say we know something about the rate of change of a function. What does this tell us about the function? And why is this important? These are the questions we will address for the remainder of the course.

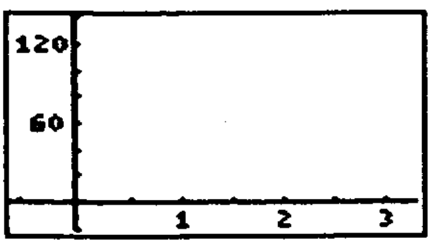
An Introductory Example

The distance from Tucson to Mesa, Arizona, is approximately 120 miles.

1. If I make this trip at a constant velocity, and it takes me 2 hours, how fast am I traveling? ______________
Sketch a graph of my distance traveled vs. time..

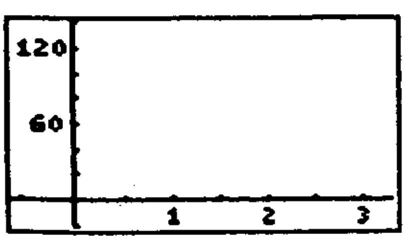
What is the equation of the distance vs. time graph above? ______________

What would the graph of the velocity vs. time graph look like? Sketch it below.



It is easy to see that the slope of the distance vs. time graph is the height of the constant function for the velocity vs. time graph. But, an interesting question is, how could the velocity vs. time graph tell the distance I traveled on the trip?

2. Suppose that instead of driving 60 mph for the entire trip, I drive 40 mph for 3/4 hr. Then I get on a highway and I drive 70 mph for 1 hr, then get off the highway and drive 40 mph again for 1/2 hr. Will I get to my destination (120 miles from Tucson) in that amount of time? Explain!



A More Interesting Example

In the preceding example, the velocity was constant over all of the time intervals. Of course, this is not always the case. We now look at an example where the velocity is continually changing.

Suppose a car is moving and accelerating over a 10 second time interval. And we calculate the car's velocity every two seconds (in ft/sec), obtaining the data shown in the table below.

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	5	10	18	31	45	65

The question is: **How far has the car traveled?**

Since we don't know how fast the car is moving at every moment, we can't calculate the distance exactly, but we can make an estimate.

Over the first two seconds the car goes <u>at least</u> ________________ feet. During the next two seconds, it goes at least ________________ feet. So, during the ten-second period the car travels at least:

Distance ≈ ____

This answer is an **underestimate** of the total distance traveled during the ten seconds. How can we obtain an overestimate of the total distance?

Distance ≈ ____

Perhaps a better estimate of the total distance traveled would be to take the average of the underestimate and the overestimate.

Distance ≈ ____________________________

A new, and important, question is: **How can we obtain a more accurate estimate of the total distance traveled?**

The data in the table below is the velocity of the car measured at every second over the ten second period.

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	5	7	10	13	18	24	31	38	45	54	65

Calculate a **lower estimate** of the total distance traveled.

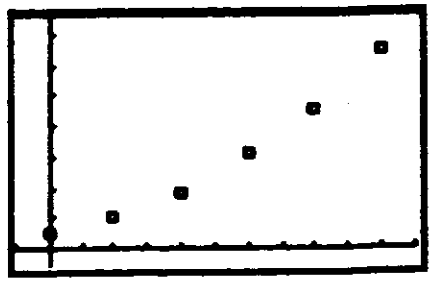
Distance = ____

Calculate an **upper estimate** of the total distance traveled.

Distance = ____

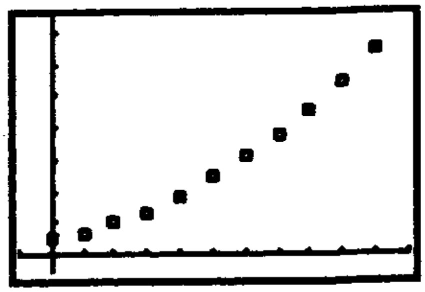
Visualizing the Distance on a Velocity Graph

Consider the two second data in the first table. The data is recorded graphically in the scatterplot shown. Draw a smooth curve through the data points.



Using the fact that the area of a rectangle is: Area = Base • Height, show the rectangles that represent the under estimate for the total distance traveled by the car. Then show the rectangles that represent the over estimate of the total distance.

Repeat the above procedure to show the rectangles that represent the underestimate and the overestimate for the data recorded every one second.



Notice that as we add more data points, the rectangles used to estimate the distance traveled fit the curve more closely. Fifty rectangles would fit the curve even more closely and give us an even better estimate for the exact distance traveled than using ten rectangles. One hundred rectangles ....

Can you see where we are going?

A Final Example

Land management officials notice that an introduced species of tree is making serious inroads into an ecosystem. In a certain area, the number of new trees per year is increasing every year. Some of the growth rates are shown below.

Year	1950	1960	1970	1980	1990
Trees/Year	337	371	408	448	493

We want to estimate the total number of new trees that have appeared between 1950 and 1990.

1. What is the minimum number of new trees that could have appeared between 1950 and 1990?

2. What is the maximum number of new trees that could have appeared between 1950 and 1990?

3. Suppose that some additional growth figures are obtained, and shown below.

Year	1955	1965	1975	1985
Trees/Year	343	389	418	467

4. Recalculate the under estimate and over estimate using this new information.

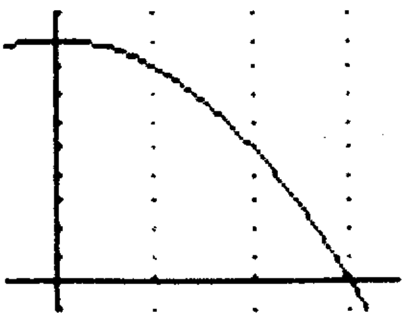
The Definition of the Definite Integral **(Section 5.2)** **Date** ______________

In the previous section we saw how to approximate the total change of a function if we were given the rate of change (represented by a graph or a table). We discovered that this total change was approximated by calculating the areas of rectangles, whose heights were determined by the rate of change function.

Today, we will continue with this discussion of finding areas of rectangles to determine an accumulated sum, but instead of calculating an "under estimate" and an "over estimate", we are going to calculate a "left-hand sum" and a "right-hand sum". Let's look at this procedure with 3 examples; the first with the function represented by a graph, the second example with the function represented by a table of values, and third, the function represented by a function rule.

Example #1: Given the graph of a function f below, approximate the area under the function from t = 0 to t = 3 with a left-hand sum, and then a right-hand sum. First let the number of rectangles (called n) in your approximation equal 3, then approximate the area with n = 6.

n = 3, so Δt = ______



Assume each "tick mark" on the graph represents one unit.

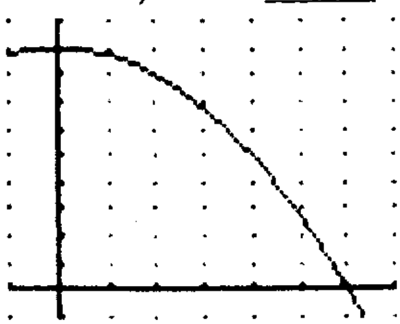
Left-hand sum = ______________________________

= ______________________________

Right-hand sum = ______________________________

= ______________________________

n = 6, so Δt = ______



The horizontal "tick marks" now represent 0.5 units.

Left-hand sum = ______________________________

= ______________________________

Right-hand sum = ______________________________

= ______________________________

Example #2: Given a function represented by the table below, approximate the left-hand sum and the right-hand sum on the interval [6 , 36].

t	6	12	18	24	30	36
f(t)	16	24	30	28	20	22

From the table, n = __________ and Δt = __________

Left-hand sum = ______________________________ = ______________

Right-hand sum = ______________________________ = ______________

Example #3: Given the function $f(t) = 2^t$, approximate the area under the function (and above the t-axis) on the interval [-2 , 6] with a left-hand sum and a right-hand sum. Let n = 4. Therefore, $\Delta t =$ ________.

Left-hand sum = ______________________________ = ______________________________ = ______

Right-hand sum = ______________________________ = ______________________________ = ______

And, once again, how could we get a better approximation for the exact area under the function?

Left-Hand Sums, Right-Hand Sums, and the Definite Integral

Suppose we have a function f(t) that is continuous for all t in the interval [a , b], and let $f(t) \geq 0$ for all t in [a , b].

We divide the interval from t = a to t = b into n equal subdivisions, each of width Δt, so

$$\Delta t = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

Let $t_0, t_1, t_2, \ldots, t_n$ be the endpoints of the subdivisions. (Therefore, $t_0 =$ ______ and $t_n =$ ________.)

Illustrate this by a graph below.

To evaluate the left-hand sum, we use the values of the function from the left endpoint of the interval, and for the right-hand sum, use values of the function from the right end of the interval. Therefore, we have:

Left-hand sum = ____

Right-hand sum = ____

Both the left-hand and right-hand sums can be written more compactly using sigma (or summation) notation.

The symbol Σ is a capital sigma, or Greek letter "S". It is used to represent a "Sum".

It is defined as: $a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$

Example: Write the series of numbers $1 + 4 + 9 + 16 + 25 + 36 + 49$ using sigma notation.

So, we can write

Right-hand sum $= f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + \ldots + f(t_n)\Delta t =$ ______________________

In the left hand sum, $f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_{n-1})\Delta t$, we start at i = _____ and stop at i = ________,
so we write

Left-hand sum $= f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_{n-1})\Delta t =$ ______________________

If f is a rate of change of some quantity, then the left-hand sum and the right-hand sum approximate the total change in the quantity. For most functions, the approximation is improved by ______________________

To find a better approximation for the **exact total change**, we take larger and larger values of n, and to find the **exact value** we: ____

Definition of the Definite Integral

Suppose f is continuous for $a \le t \le b$. The definite integral of f from a to b, written $\int_a^b f(t)dt$, is the limit of the left-hand or right-hand sums with n subdivisions of [a , b]. (Note: As $n \to$ ________ , $\Delta t \to$ ________).

Therefore, $\int_a^b f(t)dt =$ ______________________ and $\int_a^b f(t)dt =$ ______________________.

Each of these sums is called a ______________________, the function f is called the ______________,

and "a" and "b" are called the ______________________________

So, we have just seen how a definite integral is defined, but this does not tell us how to evaluate a definite integral.

Therefore, a new question is:

How do we calculate definite integrals? And using the function in Example #3, what is $\int_{-2}^{6} 2^t\,dt$?

1. We have already approximated it in Example #3 using left-hand and right-hand sums, and averaging them together.

Left-hand sum = 42.5 Right-hand sum = 170 $\int_{-2}^{6} 2^t\,dt \approx$ _________________

But if we want a good approximation, we need to use a lot of rectangles, and this is a lot of work!

2. We can approximate definite integrals with our calculator. The graphing calculator can compute sums for very large values of n.

This can be done one of two ways with our calculator.

i. From the Home Screen, press the MATH key and select "9:fnInt". (This stands for a "function numerical integral".)

Then to evaluate $\int_{-2}^{6} 2^t\,dt$, enter:

fnInt(____________________________________) = ________________

ii. If the function has been graphed on your calculator with the interval included, from the graph select CALC, and 7: $\int f(x)dx$, and enter the limits of integration.

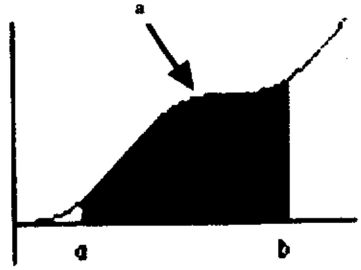
3. For some functions, we are able to calculate the definite integral exactly using geometry. (We will see this in the next class.)

4. The **Fundamental Theorem of Calculus** allows us to evaluate definite integrals exactly by hand, i.e., without our calculator, but you are going to have to wait on this!

The Definite Integral as Area **(Section 5.3)** **Date:** ____________

We have seen that if a function f(x) is continuous (no breaks) and positive (above the x-axis), then $\int_a^b f(x)dx$ represents the area under the graph of f (and above the x-axis) between x = a and x = b.

$$\text{Area} = \int_a^b f(x)dx$$



For some functions, we are able to evaluate this definite integral with geometery.

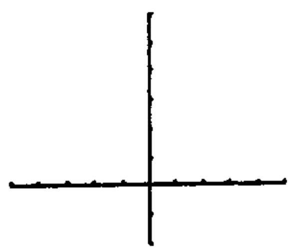
Examples:

For the following problems, sketch the region whose area is given by the definite integral. Use geometry to evaluate the integral, then check your answer with your calculator.

1. $\int_0^2 (-2x+6)dx$

Area = ____________________

2. $\int_{-3}^{3} (4-|x|)dx$



Area = ____________________

Practice Problems:

3. $\int_2^{12} (8-\frac{1}{2}x)\,dx$

4. $\int_{-3}^{3} (|x|+2)\,dx$

The next big question we need to answer is:

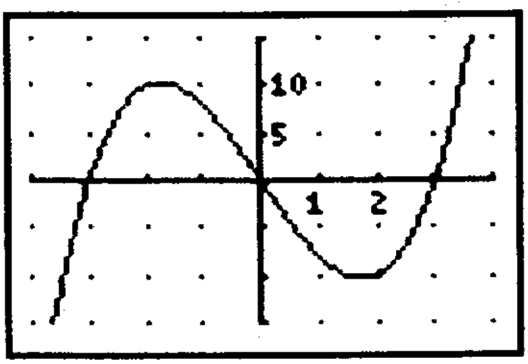
What happens to the definite integral when the function f(x) is not positive on the interval [a , b]?

Perhaps f(x) is negative on the entire interval, or, perhaps, it is positive on part of the interval and negative on another part.

Let's explore this with some examples. We also want to keep in mind the **definition of the definite integral.**

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

Below is the graph of $f(x)=x^3-9x$ from $x=-4$ to $x=4$.



Using the graph, estimate $\int_{-3}^{0}(x^3-9x)dx$.

$\int_{-3}^{0}(x^3-9x)dx \approx$ ____________

Evaluate $\int_{-3}^{0}(x^3-9x)dx$ with your calculator to see how good the guess was. $\int_{-3}^{0}(x^3-9x)dx =$ ____________

Use this result to guess, then evaluate, the following definite integrals.

1. $\int_{0}^{3}(x^3-9x)dx =$ ____________
2. $\int_{-3}^{3}(x^3-9x)dx =$ ____________
3. $\int_{0}^{-3}(x^3-9x)dx =$ ____________
4. $\int_{3}^{0}(x^3-9x)dx =$ ____________
5. $\int_{0}^{2}(x^3-9x)dx =$ ____________
6. $\int_{-3}^{2}(x^3-9x)dx =$ ____________
7. What is the total area between the function $f(x)=x^3-9x$ and the x-axis from $x=-3$ to $x=2$?

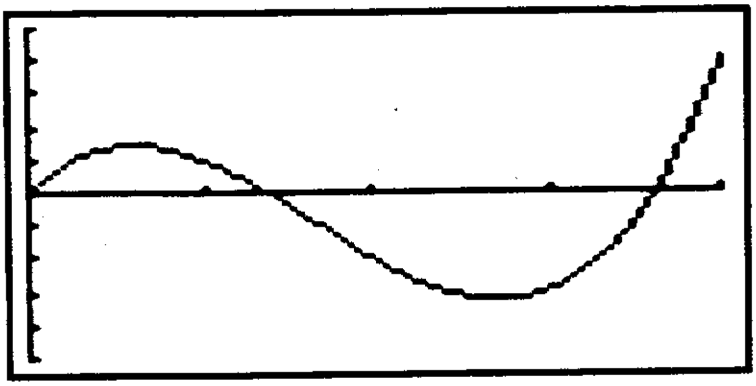
____

In conclusion,

When f(x) is positive for some x-values in the interval [a , b], and negative for other x-values in [a , b], and $a < b$,

then $\int_a^b f(x)dx$ is ____________________________________

Example: Interpret the definite integral $\int_0^4 (x^3 - 5x^2 + 5x)dx$ in terms of area. The figure below shows the graph of $f(x) = x^3 - 5x^2 + 5x$ to help in the analysis.



What is the total area between the x-axis and the function $f(x) = x^3 - 5x^2 + 5x$ from $x = 0$ to $x = 4$?

Area = ____

Interpretations of the Definite Integral **(Section 5.4)** **Date:** ___________

The objective of this section is to understand the meaning behind the notation $\int_a^b f(x)dx$, and to relate it to other applications, except just finding the "area under the curve".

First, recall the Leibniz notation for the derivative $\frac{dy}{dx}$ to represent $\frac{\Delta y}{\Delta x}$ over a "very small" interval with respect to x. In the same way $\int_a^b f(x)dx$ represents the sum of the areas of a "large number" of rectangles, each with area $f(x_i)\Delta x$. This reminds us that the definite integral is the limit of a sum. The terms that are being added together are products of the form "f(x) times a difference in x".

So what are the units of the result of calculating $\int_a^b f(x)dx$?

They are the product of ____

For example, suppose f(x) represents the rate at which the heart is pumping blood, in liters per second, where x is the time in seconds, express the units and meaning of the integral $\int_0^{10} f(x)dx$.

$\int_0^{10} f(x)dx$ ____

____

Units: ______________________

In general, if f(t) is a function that represents the **rate of change of a quantity**, then

$\int_a^b f(t)dt$ represents ____

Example 1: Assume $f(t) = 60\sqrt{t}$ gives the rate of change of the population of a city, in people per year, at time t years since 1990. If the population of the city is 3000 in 1990, what is the population in 2000?

Example 2: Suppose C(t) represents the cost per day to heat your home in dollars per day, where t is the time measured in days and $t = 0$ corresponds to January 1, 2009. Interpret the meaning of $\int_0^{90} C(t)dt$ with units.

$\int_0^{90} C(t)dt$ ____

____

Units: __________

The following two problems were taken from **Advanced Placement Calculus Exams** administered to high school calculus students. The first problem contains a trigonometric function, but it is not necessary to understan trigonometry to complete the problem!

Example 3, Question #1a from the 2004 AP Exam.
Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30$$

where F(t) is measured in cars per minute and t is measured in minutes. To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

Example 4, Question #2a, from the 2009 AP Exam.
The rate at which people enter an auditorium for a rock concert is modeled by the function R given by

$$R(t) = 1380t^2 - 675t^3 \text{ for } 0 \le t \le 2 \text{ hours}$$

where R(t) is measured in people per hour. One hundred people are in the auditorium at time $t = 0$, when the doo open, and the concert begins at time $t = 2$ when the doors close. Calculate $\int_0^2 R(t)dt$ and explain its meaning, including units.

Finding Antiderivatives **(Section 7.1)** **Date:** ______________

Up to this point in the course, one of our major problems has been

Given a function f, find the derivative f'.

We needed to be able to find the derivative of a function, before we could use it!

Today, we are going to look at the opposite problem, which also occurs in many applications.

Given the derivative f', find the function f.

or we could state this problem as:

Given a function f, find the function F, such that F' = f.

Examples:

1. If $f'(x) = 20x^4$, then $f(x) =$ ________________

A question about this answer. Is this a unique answer? ________

So, we say the antiderivative to $f'(x) = 20x^4$ is $f(x) =$ ____________________.

This solution represents a "family of functions" for the antiderivative and is called a "general antiderivative". The C is called the "constant of integration" and should be part of every answer to an antiderivative problem. (We will talk about finding "C" later.)

2. If $f(x) = x^4$, then $F(x) =$ ____

3. If $f(x) = x^{15}$, then $F(x) =$ ____

4. If $f(x) = \dfrac{1}{x^2} =$ __________, then $F(x) =$ ________________ = ____________________

5. If $f(x) = \sqrt{x} =$ __________, then $F(x) =$ ________________ = ____________________

Before we write this as a property, let's discuss some new notation.

Derivative Notation

1. If $f(x) = x^2$, then $f'(x) =$ ________

2. $\dfrac{d}{dx}\left(x^2\right) =$ __________

Antiderivative Notation

1. If $f(x) = x^2$, then $F(x) =$ ________________

2. $\int x^2 dx =$ ____________________

This is called an **indefinite integral.**

So, in general, $\int x^n dx =$ ____________________. This is called the **Power Rule for Antiderivatives.**

But be careful! What happens if n = -1? $\int \frac{1}{x}dx = \int x^{-1}dx =$ ____________________

____

We will handle this later!

Let's find some more antiderivatives, and, from the results, perhaps obtain some properties of indefinite integrals.

Examples and Properties: (Assume k is a constant)

1. $\int 7\,dx =$ ____________________ **Property**: $\int k\,dx =$ ____________________

2. $\int 5x^7 dx =$ ____________________ **Property**: $\int (k \cdot f(x))dx =$ ____________________

3. $\int \left(2 - 3x^2 + x^5\right)dx =$ ____________________ **Property**: $\int \left(f(x) \pm g(x)\right)dx =$ ____________________

$=$ ____________________

Note: Every derivative property has a corresponding antiderivative property!

4. $\int e^x dx =$ ____________________

5. $\int e^{4x} dx =$ ____________________ **Property**: $\int e^{kx} dx =$ ____________________

There is one more antiderivative property we need. What is $\int \frac{1}{x}dx = \int x^{-1}dx$? As we have seen, we can't apply t

Power Rule because ____

So, we are looking for an antiderivative of $\frac{1}{x}$. Fortunately we know a function whose derivative is $\frac{1}{x}$.

$$\frac{d}{dx}(\underline{\qquad\qquad}) = \frac{1}{x}.$$

But, this is only true if $x > 0$, since $\ln x$ is only defined for $x > 0$. What happens if $x < 0$?

If $x < 0$, then $\ln x$ ______________________________, so $\frac{d}{dx}(\ln x)$ ______________________________

But, if $x < 0$, then $\ln(-x)$ is defined, and

$$\frac{d}{dx}(\ln(-x)) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

This means that if $x > 0$, then $\int \frac{1}{x} dx = \ln x + C$, and if $x < 0$, then $\int \frac{1}{x} dx = \ln(-x) + C$.

Can we combine these two so that we are always taking the ln of a positive number? **Yes**, let the answer to the integral be :

ln __________ +C.

So, the last antiderivative property says:

Property: $\int \frac{1}{x} dx =$ ________________________

One final topic. We know that the **general antiderivative** of

$f(x) = \frac{3}{x} - 4x$ is $F(x) =$ ____

What would we need to find the **specific antiderivative**; i.e. what do we need to find the value of C?

So, if we were given that $f(x) = \frac{3}{x} - 4x$, and $F(1) = -10$, where F is the antiderivative of f, find the **specific antiderivative** of f(x)?

Therefore, if $f(x) = \frac{3}{x} - 4x$ and $F(1) = -10$, then $F(x) =$ ___________________________________

Example: If $f(x) = e^{2x} - 9x^2$ and $F(0) = 4$, find F(x), the specific antiderivative of f(x).

Finding Antiderivatives by the "Substitution Method" **(Section 7.2)** **Date:** ____________

In the last class we learned how to find antiderivatives of some functions. We learned that:

1. $\int \frac{1}{\sqrt{x}} dx =$

2. $\int 4e^{5x} dx =$

3. $\int \left(5 + \frac{7}{x}\right) dx =$

Today, we want to increase the types of functions for which we can find antiderivatives.

Before we get into this "method" for finding antiderivatives, we need to understand a few basic concepts.

1. If $y = f(x)$, then $\frac{dy}{dx} =$ ________, or we can write $dy =$ ____________________
   Note: "dy" is called the <u>differential of y.</u>

2. If $w = g(x)$, the $dw =$ ____________________

3. If $\int f(x)dx = h(x) + C$, then $\int f(w)dw =$ ________________________

Which antiderivative appears to be easier to find?

1. $\int (x^3 + 1)^6 dx$ 2. $\int 3x^2 (x^3 + 1)^6 dx$

To answer this question, we need to find a function such that:

1. $\frac{d}{dx}($____________$) = (x^3 + 1)^6$ or 2. $\frac{d}{dx}($____________$) = 3x^2(x^3 + 1)^6$

The result of the second problem is:

$\int 3x^2 (x^3 + 1)^6 dx =$ ______________________________

Note: The only way to find the first antiderivative, $\int (x^3+1)^6\,dx$, is to expand the integrand, and write it as:

$(x^3+1)^6 = x^{18} + 6x^{15} + 15x^{12} + 20x^9 + 15x^6 + 6x^3 + 1$ and then find the antiderivative of each term!

Another example: Since $\frac{d}{dx}\left(e^{x^2}\right) =$ ______________, we can write ______________________________

So, our objective is to be able to find an antiderivative of a function such as $\int 2x \cdot e^{x^2}\,dx$.

This results in a technique for finding antiderivatives called the **Substitution Method**. (It is the **Chain Rule** for derivatives in reverse.) For an indefinite integral such as:

$\int 2x \cdot e^{x^2}\,dx$, the idea is to look for an "inside function". For this example the inside function is ______

So, we make a variable substitution and let w = __________. This means dw = ____________________.

We then express the integrand in terms of w.

$\int 2x \cdot e^{x^2}\,dx =$ ____________________

The next step is to find this antiderivative of the function written in terms of w.

And then, we rewrite the answer back in terms of x.

Therefore, $\int 2x \cdot e^{x^2}\,dx =$ __________________________

Finally, you can always check your answer by finding it's derivative!

Let's try another example.

Find $\int 4x^3\sqrt{x^4+5}\,dx$

Let's make a small change to the above problem. Suppose we are asked to find

$$\int x^3\sqrt{x^4+5}\,dx$$

Some more examples.

Find the following antiderivatives.

1. $\int \frac{x}{4x^2+1}dx$

2. $\int \frac{1}{\sqrt{3-4x}}dx$

3. $\int e^{-5x}dx$

4. $\int x^2\sqrt{x^3-10}\,dx$

Note: If problem #4 above said $\int x\sqrt{x^3-10}\,dx$, we would not be able to find this antiderivative by the **Substitution Method**!

So, to summarize the **Substitution Method** for finding antiderivatives,

1. We make a variable substitution and let w equal an "inside function".
2. We then find dw, and see if the Substitution Method applies.
3. If it does, we then express the integrand in terms of w.
4. We then find this antiderivative of the function expressed in terms of w.
5. Finally, we rewrite the answer back in terms of x.

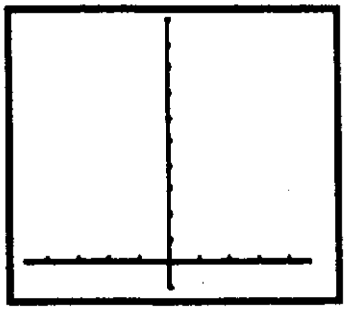
It's now time for you to practice!

Evaluating Definite Integrals **(Section 7.3)** **Date: ______________**

The Fundamental Theorem of Calculus

An introductory example:

Find the area of the region bounded by the function $f(x) = 9 - x^2$ and the x-axis. Sketch the region below.



Area of the region = ________________________________

Could we calculate this area with geometry? __________

Actually, we can! Archimedes discovered that the "area under a parabola" can be found by the formula:

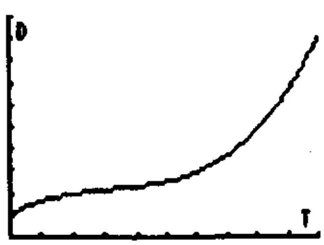
$$\text{Area} = \frac{2}{3} \cdot \text{Base} \cdot \text{Height}.$$

So, the area of the region is ________________________________

But the objective of this lesson, is to discover how to evaluate definite integrals analytically, without our calculator. This result is called the **Fundamental Theorem of Calculus** (FTC).

Let's develop the FTC through an "application".

Honors Physics student Kal Kulis is performing an experiment. He has set a motion detector on a table and it is recording his distance in feet from the motion detector as he is walking away. He collects this data over a 10 second period, enters it into his calculator, and graphs it. The result is shown below.



The graph shows a Distance vs. Time graph. Kal, being also an excellent mathematics student, wants to find a function that models the graph. He decides that the data might be modeled by a cubic function, so he performs a cubic regression on his calculator, and gets a perfect fit!

His function, which represents his distance s(t) from the motion detector at any time t is:

$$s(t) = \frac{1}{10}t^3 - t^2 + 4t + 5$$

Questions:

1. How far did Kal walk over the 10 second time interval? ________________________________

2. Recall that a velocity function is the derivative of a distance function. Sketch a graph of the velocity function below.

3. What is the velocity function for Kal's distance function $s(t) = \frac{1}{10}t^3 - t^2 + 4t + 5$?

v(t) = ____ (This is the function we graphed in part 2.)

4. Since the velocity function is a "rate of change" function, how could we use it to determine how far Kal walked over the 10 second time interval?

____

Let's calculate it (with our calculator) and see what we get! ________________________________

So, here's what we just discovered.

____

And, since the velocity v(t) is the derivative of the distance s(t), then the distance s(t) is the ______________ of the velocity v(t).

Let's generalize this result.

If F(x) is the antiderivative of f(x), and we are asked to evaluate $\int_a^b f(x)dx$, we can do this by writing

$\int_a^b f(x)dx =$ ____

This is called the **Fundamental Theorem of Calculus**, and we can use it to evaluate definite integrals, if we know how to find the antiderivative of the integrand.

Let's evaluate the definite integral $\int_{-3}^{3}(9-x^2)dx$, which we used in the "introductory example" with the FTC and see if we get the same answer (which was 36).

$\int_{-3}^{3}(9-x^2)dx =$

Some more examples:

Evaluate the following definite integrals with the FTC.

1. $\int_4^9 \sqrt{x}\,dx$

2. $\int_1^2 \frac{3}{x^4}dx$

3. $\int_0^1 \left(5e^x + 6x\right)dx$

4. $\int_1^e \frac{1}{t}dt$

In conclusion, The **Fundamental Theorem of Calculus** states:

If f(x) is a continuous function on the interval [a , b], and F(x) is the antiderivative of f(x),

then $\int_a^b f(x)dx$ = ____

We have actually used this earlier, when we did **Section 5.4: Interpreting the Definite Integral.**

Example #1: Assume $f(t) = 60\sqrt{t}$ gives the rate of change of the population of a city, in people per year, at time . years since 1990. If the population of the city is 3000 in 1990, what is the population in 2000?

The FTC with the Substitution Method **(Section 7.3 continued)** **Date: __________**

The last two sections presented in the course were:

1. Finding antiderivatives with the Substitution Method.
2. Evaluating definite integrals using the Fundamental Theorem of Calculus

Today, the last day(!), we are going to combine these two topics. In other words, how do we use the FTC to evaluate a definite integral if it is necessary to use the Substitution Method to find the antiderivative.

There are two methods that we can use to do this. Let's look at an example.

Example:

Evaluate: $\int_0^1 x\left(x^2+1\right)^3 dx$

Method #1: To evaluate this definite integral using the FTC, we first need to find an antiderivative of the function $f(x)=x\left(x^2+1\right)^3$, which requires the Substitution Method. So, let's first treat the problem as an indefinite integral (antiderivative), and find:

$$\int x\left(x^2+1\right)^3 dx =$$

Now, that we have the antiderivative, we can find the definite integral.

$$\int_0^1 x\left(x^2+1\right)^3 dx =$$

Note: Check the answer with your calculator.

Method #2: Once we have established the "change of variable" in the definite integral, and we have written the antiderivative in terms of the new variable, we can also rewrite the limits of integration in terms of the new variable.

$$\int_0^1 x\left(x^2+1\right)^3 dx$$

With this method, we do not convert the antiderivative back to the original variable.

Let's try some more examples to get some practice.

Examples and Practice Problems:

Analytically, evaluate the following definite integrals. You can then check answers with your calculator.

1. $\int_0^2 x^2\sqrt{1+x^3}\,dx$

2. $\int_0^4 \frac{dx}{\sqrt{2x+1}}$

3. $\int_1^2 \frac{dx}{(3-5x)^2}$

4. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

5. $\int_0^3 \frac{6x}{x^2+1}dx$

6. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = 460e^{1.2t}$ bacteria per hour. How many bacteria will there be after 3 hours?